

PATTERN-EQUIVARIANT HOMOLOGY

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ABSTRACT. Pattern-equivariant (PE) cohomology is a well-established tool with which to interpret the Čech cohomology groups of a tiling space in a highly geometric way. In this paper we consider homology groups of PE infinite chains. A generalised setting in which to consider PE homology and cohomology is established. We prove Poincaré duality between the two under certain conditions, which applies to examples such as the translational hull of an FLC tiling. So PE chains may be used to visualise topological invariants of tilings. The PE homology groups for a Euclidean tiling based upon chains which are PE with respect to the group of orientation preserving rigid motions exhibit a singular behaviour at points of rotational symmetry, which often adds extra torsion to the calculated invariants. We present an efficient method of computation of the PE (co)homology groups for hierarchical tilings.

INTRODUCTION

Periodic patterns of Euclidean space have been of great important to mathematics, and aesthetics, for centuries. However, the geometries of periodic patterns are well understood, and there is a rich class of patterns, such as the Penrose tilings, which exhibit intricate internal structure without possessing translational symmetry. Aperiodic tilings enjoy connections with areas of mathematics such as mathematical logic [24]—as established by Berger’s proof of the undecidability of the domino problem [7], Diophantine approximation [2, 8, 17, 18], the structure of attractors [10] and symbolic dynamics [35], but also to physical applications, most notably to solid state physics in the wake of the discovery of quasicrystals by Shechtman et al. [36].

A full understanding of a periodic tiling, modulo locally defined reversible redecorations, amounts to an understanding of its symmetry group. In the aperiodic setting, the complexity and incredible diversity of examples demands a multifaceted approach. Techniques from the theory of groupoids [6], semigroups [21], C^* -algebras [1], dynamical systems [11, 20], ergodic theory [30] and shape theory [10] find natural rôles in the field, and of course these tools have tightly knit connections to each other [22]. One approach to studying a given aperiodic tiling T is to associate to it a moduli space Ω_T , sometimes called the tiling space, of locally indistinguishable tilings; see Sadun’s book [34] for an accessible introduction to the theory. A central goal is then to formulate methods of computing topological invariants of Ω_T , and to describe what these invariants actually tell us about the original tiling T . An important perspective, particularly for the latter half of this objective, is provided by Kellendonk and Putnam’s theory of pattern-equivariant (PE) cohomology [19, 23]. The PE cohomology groups allow for an intuitive geometric description of the Čech cohomology $\check{H}^\bullet(\Omega_T)$ of the tiling space. Over \mathbb{R} coefficients the PE cochain groups may be defined using PE differential forms

[19], and over general Abelian coefficients, when the tiling has a cellular structure, with PE cellular cochains [33].

In this paper we introduce the *pattern-equivariant homology groups* of a tiling. The PE chain complex is essentially defined by replacing the cellular coboundary maps of the PE cellular cochain complex with the cellular boundary maps. So the elements of the PE chain groups are based on infinite cellular chains (sometimes called Borel–Moore chains) which respect the internal symmetries of the pattern.

Although the PE homology groups may be defined using singular chains without reference to a specific cellular structure for the underlying pattern [38], in this exposition we shall avoid the singular theory and work exclusively with cellular chains. We review the combinatorial theory of regular CW complexes in §1, which one may think of as a generalisation of the theory of abstract simplicial complexes. This allows us, in §2, to introduce the notion of a *system of internal symmetries* (SIS, for short) over a regular CW complex. Constructions such as the PE cochain complex or the inverse limit of approximants associated to a cellular tiling extend to these objects. The setting applies naturally to Euclidean as well as non-Euclidean examples, such as hyperbolic tilings or Bowers and Stephenson’s conformal tilings [9], and even to more abstract examples not canonically described by the internal symmetries of a geometric tiling of space, such as those associated to solenoids (see Example 3.9 and §4.5.5). Whilst SIS’s are introduced primarily as a convenient notational tool in which to frame our arguments, we believe that the notion should be of particular use in defining invariants for tilings in the presence of non-trivial isotropy.

In §3 we define the PE homology and cohomology groups of an SIS, and show their invariance under barycentric subdivision. We then relate the two through the following Poincaré duality result:

Theorem 3.11 *Let \mathfrak{T} be an SIS defined over an oriented homology d -manifold. If the internal symmetries of \mathfrak{T} are orientation preserving then we have PE Poincaré duality $H^\bullet(\mathfrak{T}; \mathbb{R}) \cong H_{d-\bullet}(\mathfrak{T}; \mathbb{R})$ over divisible coefficients. Moreover, if \mathfrak{T} has trivial local isotropy, then we have PE Poincaré duality $H^\bullet(\mathfrak{T}) \cong H_{d-\bullet}(\mathfrak{T})$ over general Abelian coefficients.*

For example, we have Poincaré duality between the Čech cohomology of the translational hull of an FLC tiling T of \mathbb{R}^d and the PE homology of T with respect to translations.

So the PE homology groups provide highly geometric descriptions of pre-established topological invariants associated to aperiodic tilings; see, for example, Figure 4.4. When one wishes to incorporate rotational symmetries, so that \mathfrak{T} may possess non-trivial local isotropy, the PE Poincaré duality of Theorem 3.11 generally fails over non-divisible coefficients and the PE homology provides a new invariant. The PE homology often picks up extra torsion elements to the PE cohomology. Although we demonstrate in §3.4 how one may modify the PE homology groups so as to regain duality, we consider this extra torsion as a phenomenon of potential interest. Indeed, in forthcoming work [37] (see also [38]) we will show how this extra torsion may be incorporated into a

spectral sequence converging to the Čech cohomology $\check{H}^\bullet(\Omega_T^{\text{rot}})$ of the Euclidean hull of a 2-dimensional tiling.

In §4 we present a method of computation for the PE homology of a hierarchical tiling, along with some worked through examples. The method is applicable to a broad range of tilings, including Euclidean ‘mixed substitution tilings’ (see [15]) but also non-Euclidean examples, such as Bowers and Stephenson’s combinatorial pentagonal tilings (see [9] and §4.5.6). The ‘approximant homology groups’ of the computation and the ‘connecting maps’ between them have a direct description in terms of the combinatorics of the star-patches. In [16], Gonçalves used the duals of these approximant chain complexes for a computation of the K -theory of the C^* -algebra of the stable equivalence relation of a substitution tiling. Our method of computation of the PE homology groups seems to confirm the observation there of a certain duality between these K -groups and the K -theory of the tiling space.

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1. PRELIMINARIES

1.1. Regular CW Complexes. We shall denote by E^n the open unit disc, by B^n the closed unit disc and by S^{n-1} the unit $(n-1)$ -sphere of \mathbb{R}^n . An *open n -cell* of a topological space X is a subspace $e \subseteq X$ which is homeomorphic to E^n .

Definition 1.1. Let X be a Hausdorff topological space and $\mathcal{K} = \{e_i \mid i \in \mathcal{I}\}$ be a partition of X as a disjoint union of open cells e_i . Denote the pair of X and \mathcal{K} by $X_{\mathcal{K}}$, and the n -skeleton $X_{\mathcal{K}}^n \subseteq X$ by

$$X_{\mathcal{K}}^n := \coprod_{i \in \mathcal{I}: \dim(e_i) \leq n} e_i.$$

Then $X_{\mathcal{K}}$ is called a *CW complex* if:

- (1) For $e_i \in \mathcal{K}$ there exists a continuous map (called a *characteristic map*) $f: B^n \rightarrow X$ which restricts to a homeomorphism $f|_{E^n}: E^n \rightarrow e_i$ and is such that $\text{im}(f|_{S^{n-1}}) \subseteq X_{\mathcal{K}}^{n-1}$.
- (2) For $e_i \in \mathcal{K}$ the closure $\overline{e_i}$ intersects only finitely many elements of \mathcal{K} .
- (3) A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e_i}$ is closed in X for all $e_i \in \mathcal{K}$.

Whilst CW complexes are ‘nice’ topological spaces which may be constructed by inductively attaching cells, there are not many constraints imposed on the characteristic maps governing how cells may be attached. In contrast, for a space with a *triangulation*, its topology is totally described up to homeomorphism by the combinatorial data of the simplexes and their incidences, i.e., the data of which simplexes are faces of others. Simplicial complexes are too restrictive to be practical as a starting point for us here; most well-known tilings are naturally cellular but not simplicial. A good middle-ground between the versatility of CW complexes and the combinatorial nature of simplicial complexes seems to be the setting of *regular* CW complexes:

Definition 1.2. A CW complex $X_{\mathcal{K}}$ is called *regular* if its characteristic maps may be chosen to be homeomorphisms.

For a CW complex $X_{\mathcal{K}}$ and open n -cell $e_i \in \mathcal{K}$, the closure $\overline{e_i}$ of e_i in X will be called a *closed n -cell* of $X_{\mathcal{K}}$. In the case that $X_{\mathcal{K}}$ is regular, each closed n -cell is homeomorphic to the closed n -disc B^n . Note, however, that the closed cells being homeomorphic to n -discs does not guarantee that a CW complex is regular.

1.2. CW Posets. We shall now justify the assertion that regular CW complexes are combinatorial in character, and in so doing set up some important notation for the remainder of the paper. The main idea (see for example Björner [5]) is that one may associate to a regular CW complex $X_{\mathcal{K}}$ a *face poset* which uniquely identifies $X_{\mathcal{K}}$ up to cellular homeomorphism.

Definition 1.3. Let $X_{\mathcal{K}}$ be a CW complex. We define the *face poset* $\mathcal{F}(\mathcal{K})$ by setting the underlying set of $\mathcal{F}(\mathcal{K})$ to be the set of closed cells of $X_{\mathcal{K}}$ and let $a \leq b$ in $\mathcal{F}(\mathcal{K})$ if and only if $a \subseteq b$.

The above allows one to associate to any CW complex a poset. Conversely, given any poset, there is a natural way to associate to it an abstract simplicial complex, and hence a topological space:

Definition 1.4. For a poset \mathcal{P} , we define its *order complex* \mathcal{P}_{Δ} as follows. The underlying set of \mathcal{P}_{Δ} is given by the set of *finite chains* of finite, non-empty, linearly ordered subsets of \mathcal{P} (i.e., subsets $\{a_1, \dots, a_n\}$ where $a_i < a_{i+1}$). For two such chains A, B , we set $A \leq B$ in \mathcal{P}_{Δ} if $A \subseteq B$.

Definition 1.5. An *abstract simplicial complex* is a non-empty set V (called the *vertex set*) together with a collection \mathcal{S} of non-empty subsets of V for which:

- (1) $\{v\} \in \mathcal{S}$ for all $v \in V$.
- (2) If $B \in \mathcal{S}$ and $\emptyset \neq A \subseteq B$, then $A \in \mathcal{S}$ also.

Clearly the order complex \mathcal{P}_{Δ} of any poset \mathcal{P} is an abstract simplicial complex with vertex set the underlying set of \mathcal{P} . To an abstract simplicial complex \mathcal{S} one may associate a topological space $|\mathcal{S}|$, called the *geometrical realisation* of \mathcal{S} . The following shows that a regular CW complex is determined, up to cellular homeomorphism, by the combinatorics of its closed cells and their incidences (c.f., Björner [5, §3], Lundell and Weingram [25, Theorem 1.7] or Massey [26, Chapter IX §6]):

Proposition 1.6. Let $X_{\mathcal{K}}$ be a regular CW complex. Then there exists a homeomorphism $h: |\mathcal{F}(\mathcal{K})_{\Delta}| \rightarrow X$. Moreover, h may be chosen so that:

- (1) For a closed cell a of \mathcal{K} , the subspace of $|\mathcal{F}(\mathcal{K})_{\Delta}|$ corresponding to the set of simplexes $\{\{a_0, \dots, a_n\} \in \mathcal{F}(\mathcal{K})_{\Delta} \mid a_n \leq a\}$ is mapped by h onto a .
- (2) For an open cell $e \in \mathcal{K}$, the subspace of $|\mathcal{F}(\mathcal{K})_{\Delta}|$ corresponding to the set of simplexes $\{\{a_0, \dots, a_n\} \in \mathcal{F}(\mathcal{K})_{\Delta} \mid a_n = \overline{e}\}$ is mapped by h onto e .

So a regular CW complex may be subdivided into a simplicial complex in a way which respects the structure of the original cell decomposition. It follows from these considerations that there is effectively no loss of information in passing from a regular CW

complex $X_{\mathcal{K}}$ to its face poset $\mathcal{F}(\mathcal{K})$. We shall embrace this philosophy in what follows by working chiefly with face posets of regular CW complexes, which shall frequently simplify our notation and arguments. A poset which is isomorphic to the face poset of a regular CW complex is called a *CW poset*. We shall refer to the order complex \mathcal{F}_{Δ} of a CW poset \mathcal{F} as *the barycentric subdivision of \mathcal{F}* . We denote by $|\mathcal{F}|$ its geometric realisation, which by the above one may unambiguously define, up to homeomorphism, to be $|\mathcal{F}_{\Delta}|$ or as any space X possessing a regular CW decomposition \mathcal{K} with $\mathcal{F}(\mathcal{K}) \cong \mathcal{F}$. Elements $a \in \mathcal{F}$ of a CW poset shall be called *cells*. If $a \leq b$ then we shall say that a is a *face* of b , and that b is a *coface* of a . For $a \in \mathcal{F}$, its dimension may be defined by the cardinality (minus one) of any maximal chain $\{a_0, \dots, a_{\dim(c)} = a\}$ terminating in a (note that we do not include the ‘empty face’ here). We shall allow ourselves to denote a by a^n when it has dimension n . CW posets exhibit the \diamond -property, taking its name from the corresponding Hasse diagram: for any $a^n < c^{n+2}$ there are precisely two $(n+1)$ -cells b_i^{n+1} satisfying $a^n < b_i^{n+1} < c^{n+2}$.

Definition 1.7. Let \mathcal{F} be a CW poset and $S \subseteq \mathcal{F}$. Then we define the *closure* of S to be the sub-poset

$$\overline{S} := \{a \in \mathcal{F} \mid \exists s \in S : a \leq s\}.$$

Note that \overline{S} is itself a CW poset. For $a \in \mathcal{F}$ we write \overline{a} for the *cell poset* $\overline{\{a\}}$ of faces of a . We call S a *subcomplex* if $S = \overline{S}$ (that is, if S is downwards closed).

For $a \in \mathcal{F}$ we define the (*open*) *star* of a to be the sub-poset of cells

$$\text{St}(a) := \{a' \in \mathcal{F} \mid a' \geq a\}.$$

We shall refer to $\overline{\text{St}(a)}$ as the *star complex* of a . We call \mathcal{F} *pure* (of dimension d) if $\text{St}(a)$ contains a d -cell, and no cells of strictly greater dimension, for all $a \in \mathcal{F}$.

We define the following collections of simplexes of \mathcal{F}_{Δ} :

- (1) The *closed cell* $a_{\Delta} := \{\{a_0, \dots, a_n\} \mid a_n \leq a\}$.
- (2) The *open cell* $a_{\Delta}^{\circ} := \{\{a_0, \dots, a_n\} \mid a_n = a\}$.
- (3) The *boundary* $\partial a_{\Delta} := \Delta(a) - \Delta(a)^{\circ} = \{\{a_0, \dots, a_n\} \mid a_n \preceq a\}$.
- (4) The *closed dual cell* $\hat{a}_{\Delta} := \{\{a_0, \dots, a_n\} \mid a \leq a_0\}$.
- (5) The *open dual cell* $\hat{a}_{\Delta}^{\circ} := \{\{a_0, \dots, a_n\} \mid a = a_0\}$.
- (6) The *dual boundary* $\partial \hat{a}_{\Delta} := \hat{a}_{\Delta} - \hat{a}_{\Delta}^{\circ} = \{\{a_0, \dots, a_n\} \mid a \preceq a_0\}$.
- (7) The *k-skeleton* $\mathcal{F}_{\Delta}^k := \{s \in \mathcal{F}_{\Delta} \mid s \in a_{\Delta} : \dim(a) = k\} = \{s \in \mathcal{F}_{\Delta} \mid s \in a_{\Delta}^{\circ} : \dim(a) \leq k\}$.
- (8) If \mathcal{F} is pure of dimension d , then we define the *dual k-skeleton* $\widehat{\mathcal{F}}_{\Delta}^k := \{s \in \mathcal{F}_{\Delta} \mid s \in \hat{a}_{\Delta} : \dim(a) = (d - k)\} = \{s \in \mathcal{F}_{\Delta} \mid s \in \hat{a}_{\Delta}^{\circ} : \dim(a) \geq (d - k)\}$.

Definition 1.8. Let \mathcal{F} be a CW poset. We shall say that \mathcal{F} is *finite-dimensional* if there exists some d for which $\dim(a) \leq d$ for all $a \in \mathcal{F}$. We call \mathcal{F} *locally finite* if $\text{St}(a)$ is finite for all $a \in \mathcal{F}$.

Henceforth, all CW posets considered shall be taken to be finite-dimensional and locally finite.

1.3. Homology and Cohomology of Regular CW Complexes. Given a chain complex C_\bullet , we shall denote its homology, as a \mathbb{Z} graded collection of groups, by $H(C_\bullet)$. The degree n homology group shall be denoted by $H_n(C_\bullet)$. Given a CW complex X_K one may compute the singular homology of X using instead the cellular chain complex $C_\bullet(X_K)$. As a group $C_n(X_K)$ is isomorphic to the free Abelian group generated by the set of n -cells of K . To determine the boundary maps $\partial_n: C_n(X_K) \rightarrow C_{n-1}(X_K)$, one uses the formula

$$\partial_n(b^n) = \sum_{a^{n-1}: a^{n-1} \leq b^n} [a^{n-1}, b^n] a^{n-1}$$

where the elements a^{n-1} and b^n in the above correspond to cells of K , considered as elements of $C_n(X_K)$ with their preferred orientations. One then defines ∂_n by extending from the above formula linearly. The *incidence numbers* $[a, b]$ of the above are integers determined by a choice of orientation for each cell and the characteristic maps of the CW complex. A practical motivation for using *regular* CW complexes X_K is that a choice of orientations for the cells of K determines and is determined by incidence numbers $[a, b]$ satisfying a few simple and intuitive axioms in terms of the CW poset $\mathcal{F}(K)$ (see Cooke and Finney [12, Chapter II, §1] or Massey [26, Chapter IX, §7]).

The above approach allows one to compute $H_\bullet(X_K)$ directly from $\mathcal{F}(K)$. However, since we shall need to be able to compare orientations of cells under cellular isomorphism, we still require a notion of orientation for cells. One way is to define an orientation on a cell $a^n \in \mathcal{F}$ to be a simplicial n -chain of a_Δ with coefficients $+1$ or -1 and boundary supported on the subcomplex ∂a_Δ . Such a chain is determined by choosing any chain $\{a_0, \dots, a_n = a\}$ of length $(n+1)$ and assigning it coefficient $+1$ or -1 . There are precisely two orientations of any given cell, we call these two orientations *opposite*. An orientation ω_b of a cell b defines an orientation $\omega \downarrow_a^b$ of any codimension one face a of b in a canonical way. In the case that \mathcal{F} is simplicial, one may associate to each orientation ω_{a^n} a simplicial orientation in a way which is natural with respect to restrictions to faces (recall that a simplicial orientation $\pm 1[v_0, \dots, v_n]$ is a signed and ordered list of the vertices of a^n , taken up to the relation of permutation $\pm 1[v_0, \dots, v_n] \simeq (\pm 1)^{\text{sgn}(\sigma)} \sigma[v_0, \dots, v_n]$. Such an orientation restricts to any face by omission of the corresponding vertex).

The usual cellular chain groups $C_i(X_K)$ are based on finite chains, each supported on a compact subcomplex of X_K . Finite chains will not be of much use to us in representing invariants on infinite tilings. However, for a locally finite CW complex one may define homology groups based on infinite cellular chains, sometimes known as Borel–Moore chains¹.

Definition 1.9. Let \mathcal{F} be a CW poset. We define the *cellular Borel–Moore chain complex* $C_\bullet^{\text{BM}}(\mathcal{F})$ as follows. Set $C_n^{\text{BM}}(\mathcal{F})$ to be the Abelian group of functions

$$\sigma: \{\text{oriented } n\text{-cells of } \mathcal{F}\} \rightarrow \mathbb{Z}$$

for which $\sigma(\omega_a^+) = -\sigma(\omega_a^-)$ for opposite orientations ω_a^+ and ω_a^- of a cell a . Of course, $\sigma_1 + \sigma_2$ is defined by setting $(\sigma_1 + \sigma_2)(\omega) := \sigma_1(\omega) + \sigma_2(\omega)$. We define the boundary

¹But also, at least under some mild restrictions on the CW complex, as ‘locally finite chains’ or ‘infinite chains’, and the corresponding homology may go by ‘homology with closed/compact supports’.

maps $\partial_n: C_n^{\text{BM}}(\mathcal{F}) \rightarrow C_{n-1}^{\text{BM}}(\mathcal{F})$ by setting

$$\partial_n(\sigma)(\omega_a) := \sum_{\omega_b: \omega_a = \omega \downarrow_a^b} \sigma(\omega_b)$$

and extending linearly. The homology $H_\bullet^{\text{BM}}(\mathcal{F}) := H(C_\bullet^{\text{BM}}(\mathcal{F}))$ of this chain complex will be called the *Borel–Moore homology of \mathcal{F}* .

The boundary maps are well-defined due to our running assumption that the CW posets under consideration are locally finite. By setting an orientation for each cell $a \in \mathcal{F}$, one has a canonical isomorphism

$$C_n^{\text{BM}}(\mathcal{F}) \cong \prod_{a \in \mathcal{F}: \dim(a)=n} \mathbb{Z}.$$

With this identification, the boundary maps agree on the ‘generators’ with the usual cellular boundary maps for finite cellular chains.

Given a cell a and a codimension one coface b of a , an orientation ω_a of a induces an orientation $\omega \uparrow_a^b$ of b .

Definition 1.10. Let \mathcal{F} be a CW poset. Then we define the *cellular cochain complex* $C^\bullet(\mathcal{F})$ as follows. Set $C^n(\mathcal{F})$ to be the Abelian group of functions

$$\psi: \{\text{oriented } n\text{-cells of } \mathcal{F}\} \rightarrow \mathbb{Z}$$

for which $\psi(\omega_a^+) = -\psi(\omega_a^-)$ for opposite orientations ω_a^+ and ω_a^- (so $C_n^{\text{BM}}(\mathcal{F}) = C^n(\mathcal{F})$). We define the coboundary maps $\delta^n: C^n(\mathcal{F}) \rightarrow C^{n+1}(\mathcal{F})$ by setting

$$\delta^n(\psi)(\omega_b) := \sum_{\omega_a: \omega_b = \omega \uparrow_a^b} \psi(\omega_a)$$

and extending linearly. The cohomology $H^\bullet(\mathcal{F}) := H(C^\bullet(\mathcal{F}))$ of this cochain complex will be called the *cohomology of \mathcal{F}* .

Notation. Let σ be a Borel–Moore chain of the CW poset \mathcal{F} . Given a set of cells $S \subseteq \mathcal{F}$, we shall write $\sigma \upharpoonright S$ to denote the restriction of σ to S , which we consider to be a chain of the subcomplex \overline{S} .

A *cellular isomorphism* of CW posets \mathcal{F} and \mathcal{G} is a bijection Φ between the underlying sets of \mathcal{F} and \mathcal{G} for which Φ and Φ^{-1} are order-preserving. Such a cellular isomorphism induces another $\Phi_\Delta: \mathcal{F}_\Delta \rightarrow \mathcal{G}_\Delta$ between barycentric subdivisions by setting

$$\Phi_\Delta(\{a_0, \dots, a_k\}) := \{\Phi(a_0), \dots, \Phi(a_k)\}.$$

Pushforwards of chains are denoted by $\Phi_*(\sigma)$. We shall frequently abuse notation by writing Φ_* in place of $(\Phi_\Delta)_*$. For example, given a cellular isomorphism between cell posets $\Phi: \overline{a} \rightarrow \overline{b}$ and an orientation ω_a of a , we may push forward the orientation on a to the orientation $\Phi_*(\omega_a) := (\Phi_\Delta)_*(\omega_a)$ on b ; here, we are considering the orientation ω_a as a simplicial n -cycle of the relative pair $(a_\Delta, \partial a_\Delta)$.

Recall that a *quasi-isomorphism* is a (co)chain map which induces isomorphisms between (co)homology groups.

Proposition 1.11. Let \mathcal{F} be a CW poset. Then there exist quasi-isomorphisms:

$$\begin{aligned}\iota_\bullet &: C_\bullet^{\text{BM}}(\mathcal{F}; G) \rightarrow C_\bullet^{\text{BM}}(\mathcal{F}_\Delta) \\ \iota^\bullet &: C^\bullet(\mathcal{F}_\Delta; G) \rightarrow C^\bullet(\mathcal{F})\end{aligned}$$

Proof. Whilst this proposition is standard, we shall provide the essential details of the proof which shall be useful later when restricting to pattern-equivariant subcomplexes. A chain $\sigma \in C_n^{\text{BM}}(\mathcal{F})$ canonically determines a simplicial chain $\sigma_\Delta \in C_n^{\text{BM}}(\mathcal{F}_\Delta)$ by setting $(\sigma_\Delta \lrcorner a_\Delta) = \sigma(\omega_a) \cdot \omega_a$ for any orientation ω_a (considered as a simplicial n -chain) on an n -cell a . This defines a simplicial chain supported on the n -skeleton \mathcal{F}_Δ^n whose boundary is supported on the $(n-1)$ -skeleton \mathcal{F}_Δ^{n-1} . Such chains form a sub-chain complex

$$\dots \xleftarrow{\partial_n} H_n^{\text{BM}}(\mathcal{F}_\Delta^n, \mathcal{F}_\Delta^{n-1}) \xleftarrow{\partial_{n+1}} H_{n+1}^{\text{BM}}(\mathcal{F}_\Delta^{n+1}, \mathcal{F}_\Delta^n) \xleftarrow{\partial_{n+1}} \dots$$

of $C_\bullet^{\text{BM}}(\mathcal{F}_\Delta)$. It essentially follows from the definitions that the map $(-)_\Delta$ induces a chain isomorphism from $C_\bullet^{\text{BM}}(\mathcal{F})$ to the above relative complex.

The map ι_\bullet is then induced by the canonical inclusion of the above relative complex into the chain complex $C_\bullet(\mathcal{F}_\Delta)$. Using the usual diagram chases, the result follows from the fact that the homologies of the relative pairs $H_n^{\text{BM}}(\mathcal{F}_\Delta^k, \mathcal{F}_\Delta^{k-1})$ vanish for $n \neq k$. The proof for cohomology is analogous. \square

The cellular Borel–Moore chain complexes of \mathcal{F} defined above are based on \mathbb{Z} coefficients. One may also define the chain complexes over G coefficients, where G is an arbitrary (discrete) Abelian group. We write $C_\bullet^{\text{BM}}(\mathcal{F}; G)$ for the cellular chain complex of \mathcal{F} over G coefficients, where the degree n chain group is based instead on functions

$$\sigma: \{\text{oriented } n\text{-cells of } \mathcal{F}\} \rightarrow G.$$

One may similarly define the cellular cochain complexes over G coefficients, denoted $C^\bullet(\mathcal{F}; G)$. We shall frequently drop the notation of the coefficient group where it is clear from context which coefficient group is in question.

1.4. Poincaré Duality. The classical ‘cell, dual cell proof’ of Poincaré duality, whilst usually stated in the simplicial setting (see for example Munkres [27]) follows over almost word-for-word to the regular CW setting, which we shall briefly recall here.

Definition 1.12. Let \mathcal{S} be a simplicial complex which possesses a partial ordering on its vertex set which linearly orders the vertices of each simplex. Then we define a homomorphism

$$\cap: C^p(\mathcal{S}) \otimes C_{p+q}^{\text{BM}}(\mathcal{S}) \rightarrow C_p^{\text{BM}}(\mathcal{S})$$

by setting

$$\psi^p \cap ([v_0, \dots, v_{p+q}]) := [v_0, \dots, v_q] \cdot \psi^p([v_q, \dots, v_{p+q}])$$

and extending linearly. Here, simplicial chains are written with the elements in reverse order, so that $v_0 > v_1 > \dots > v_{p+q}$.

One may easily check that the cap product satisfies

$$\partial(\psi^p \cap \sigma_{p+q}) = (-1)^q(\delta(\psi^p) \cap \sigma_{p+q}) + \psi^p \cap \partial(\sigma_{p+q}).$$

It follows that the cap product induces a homomorphism

$$\cap: H^p(\mathcal{S}) \otimes H_{p+q}^{\text{BM}}(\mathcal{S}) \rightarrow H_q^{\text{BM}}(\mathcal{S}).$$

Let \mathcal{F} be a pure d -dimensional CW poset and G be some coefficient ring. Given a choice of orientation of each d -cell of \mathcal{F} , there is an associated Borel–Moore d -chain given by assigning coefficient $+1$ to each preferred orientation. We shall say that \mathcal{F} is *oriented* if it is equipped with a Borel–Moore cycle Γ , called a *fundamental class*, associated to some choice of orientation of each d -cell.

One may canonically identify the chain complex $C_{\bullet}^{\text{BM}}(\mathcal{F})$ with the sub-chain complex of $C_{\bullet}^{\text{BM}}(\mathcal{F}_{\Delta})$ which in degree n consists of simplicial n -chains supported on \mathcal{F}_{Δ}^n and with boundaries supported on $\mathcal{F}_{\Delta}^{n-1}$. In a similar way we may define the *dual chain complex* $C_{\bullet}^{\text{BM}}(\widehat{\mathcal{F}})$:

Definition 1.13. Let \mathcal{F} be a pure CW poset. Then we define the *dual chain complex* $C_{\bullet}^{\text{BM}}(\widehat{\mathcal{F}}; G)$ to be the sub-chain complex of $C_{\bullet}^{\text{BM}}(\mathcal{F}_{\Delta}; G)$ which in degree n consists of simplicial n -chains supported on the dual n -skeleton $\widehat{\mathcal{F}}_{\Delta}^n$ and with boundaries supported on the dual $(n-1)$ -skeleton $\widehat{\mathcal{F}}_{\Delta}^{n-1}$.

Definition 1.14. A *homology d -manifold* is a locally compact topological space X for which its local singular homology groups $H_i(X, X - \{x\})$, for any $x \in X$, are trivial for $i \neq d$ and are isomorphic to \mathbb{Z} for $i = d$.

Of course, it is instructive to think of the dual chain complex as a cellular chain complex associated to the ‘dual cell decomposition’. It is worth reiterating, however, that even for a triangulated d -manifold, the dual cell decomposition need not be a CW complex.

Theorem 1.15 (Poincaré Duality). Let \mathcal{F} be a CW poset for a homology d -manifold $|\mathcal{F}|$. Fix a coefficient ring G and suppose that $C_d^{\text{BM}}(\mathcal{F}; G)$ possesses a fundamental class Γ . Then:

- (1) The cap product with the fundamental class induces a cochain isomorphism $-\cap \Gamma: C^{\bullet}(\mathcal{F}; G) \rightarrow C_{d-\bullet}^{\text{BM}}(\widehat{\mathcal{F}}; G)$.
- (2) The canonical inclusion $\iota: C_{\bullet}^{\text{BM}}(\widehat{\mathcal{F}}; G) \rightarrow C_{\bullet}^{\text{BM}}(\mathcal{F}_{\Delta}; G)$ is a quasi-isomorphism.

Hence, there exists an isomorphism $H^{\bullet}(\mathcal{F}; G) \cong H_{d-\bullet}^{\text{BM}}(\mathcal{F}; G)$.

For concreteness and later reference we shall make precise how the map $-\cap \Gamma$ in the above is explicitly defined. For a cochain $\psi \in C^n(\mathcal{F}; G)$ one may define a simplicial cochain $\psi_{\Delta} \in C^n(\mathcal{F}_{\Delta}^n, \mathcal{F}_{\Delta}^{n-1}; G)$ which, for an orientation ω_c of n -cell c , evaluates on the corresponding sum of oriented n -simplexes the value $\psi(\omega_c)$. Similarly (as in the proof of Proposition 1.11) to a chain $\sigma \in C_n^{\text{BM}}(\mathcal{F}; G)$ one may associate a simplicial chain $\sigma_{\Delta} \in C_n^{\text{BM}}(\mathcal{F}_{\Delta}; G)$. Note that the barycentric subdivision \mathcal{F}_{Δ} inherits a natural partial order on its vertices from \mathcal{F} . Then $\psi \cap \Gamma$ is defined to be $\psi_{\Delta} \cap \Gamma_{\Delta} \in C_{d-n}^{\text{BM}}(\widehat{\mathcal{F}})$. The work in showing that this induces the required isomorphism is in showing that the dual n -cells modulo their boundaries have simplicial homology in degree n freely generated by a relative fundamental class. For 2 one needs to show that the relative homology groups of the neighbouring levels of the skeleta of the dual cell decomposition vanish in the expected degrees, in analogy to a CW decomposition.

2. SYSTEMS OF INTERNAL SYMMETRIES

A tiling possessing few global symmetries may be far from totally random. There exist interesting examples of tilings—the most famous being the Penrose tilings—which, despite having no translational symmetries, have the property that any given finite motif of the pattern, no matter how large, may be found with only bounded gaps across the entire pattern. This is a property known as recurrence or repetitivity. So, without possessing any *global* symmetries, a tiling may possess a rich structure of *internal* symmetries between finite portions of it. Forgetting the precise decoration in question, the essential structure of a given tiling is recorded by a system of pseudogroups or groupoids keeping track of which points of the pattern are equivalent to a certain radius and by which local morphisms.

These ideas motivate the central definition of this section, given below. Recall that a poset (Λ, \leq) is called a *directed set* if for all $\lambda_1, \lambda_2 \in \Lambda$ there exists some $\lambda \in \Lambda$ with both $\lambda_1, \lambda_2 \leq \lambda$. We shall often say that a property holds *for sufficiently large* λ , which shall always mean that there exists some $\lambda' \in \Lambda$ for which the property holds for any $\lambda \geq \lambda'$.

Definition 2.1. A *system of internal symmetries* (or *SIS*, for short) \mathfrak{T} consists of the following data:

- A CW poset \mathcal{T} .
- A directed set (Λ, \leq) called the *radius poset*.
- For each $\lambda \in \Lambda$ and each pair of cells $a, b \in \mathcal{T}$ a set $\mathfrak{T}_{a,b}^\lambda$ of cellular isomorphisms $\Phi: \overline{\text{St}(a)} \rightarrow \overline{\text{St}(b)}$ sending a to b . We denote the collection of all such morphisms by \mathfrak{T}^λ .

This data is required to satisfy the following:

- (G1) For all $\lambda \in \Lambda$ and $a \in \mathcal{T}$ we have that $\text{Id}_{\overline{\text{St}(a)}} \in \mathfrak{T}_{a,a}^\lambda$.
- (G2) For all $\lambda \in \Lambda$ and $\Phi \in \mathfrak{T}_{a,b}^\lambda$ we have that $\Phi^{-1} \in \mathfrak{T}_{b,a}^\lambda$.
- (G3) For all $\lambda \in \Lambda$, $\Phi_1 \in \mathfrak{T}_{a,b}^\lambda$ and $\Phi_2 \in \mathfrak{T}_{b,c}^\lambda$, we have that $\Phi_2 \circ \Phi_1 \in \mathfrak{T}_{a,c}^\lambda$.
- (Inc) For all $\lambda_1 \leq \lambda_2$ we have that $\mathfrak{T}^{\lambda_1} \supseteq \mathfrak{T}^{\lambda_2}$.
- (Res) For all $\lambda \in \Lambda$ there exists some $\lambda_{\text{res}} \geq \lambda$ satisfying the following. Given any $b \in \mathcal{T}$ and face a of b , every morphism of $\mathfrak{T}_{a,-}^{\lambda_{\text{res}}}$ restricts to a morphism of $\mathfrak{T}_{b,-}^\lambda$.
- (CoRes) Dually, for all $\lambda \in \Lambda$ there exists some $\lambda_{\widehat{\text{res}}}$ satisfying the following. Given any $a \in \mathcal{T}$ and coface b of a , every morphism of $\mathfrak{T}_{b,-}^{\lambda_{\widehat{\text{res}}}}$ is a restriction of some morphism of $\mathfrak{T}_{a,-}^\lambda$.

The CW poset \mathcal{T} is to be thought of as the underlying complex on which the tiling is supported. The radius poset (Λ, \leq) will usually correspond to $\Lambda = \mathbb{R}_{>0}$ (as a ‘radial distance’) or \mathbb{N}_0 (as a ‘combinatorial distance’), although it will sometimes be helpful to allow some more flexibility. Then a morphism $\Phi \in \mathfrak{T}_{a,b}^\lambda$ is to be thought of as recording the fact that cells a and b are equivalent to radius λ in the tiling, via a morphism which is described locally by Φ . Such morphisms should include the identity morphism and be invertible and composable, as dictated by the groupoid axioms (G1), (G2) and (G3), respectively. The inclusion axiom (Inc) simply states that if two cells are equivalent to radius λ_2 in the tiling, via morphism Φ , then they are still equivalent via Φ to

any smaller radius $\lambda_1 \leq \lambda_2$. In short, for $\lambda_1 \leq \lambda_2$ we have an inclusion of groupoids $\iota: \mathfrak{T}^{\lambda_2} \hookrightarrow \mathfrak{T}^{\lambda_1}$. The final two axioms (Res) and (CoRes) of restriction and corestriction establish a coherence between the cellular structure of \mathcal{T} and the restrictions between the various morphisms of \mathfrak{T} .

2.1. Motivating Examples.

2.1.1. *Euclidean Tilings.* A *tiling* T of \mathbb{R}^d is a collection of subsets $t \subset \mathbb{R}^d$, called *tiles*, satisfying:

- (1) The tiles of T are compact and equal to the closures of their interiors.
- (2) $\mathbb{R}^d = \bigcup_{t \in T} t$.
- (3) Distinct $t_1, t_2 \in T$ intersect on at most their boundaries.

Sometimes one may wish to endow the tiles with ‘labels’ (or ‘colours’), so as to distinguish patches of tiles which are geometrically equivalent but are not so with their labels c.f., Wang tilings of decorated unit squares of \mathbb{R}^2 .

The collection of closed d -cells of a regular CW decomposition \mathcal{T} of \mathbb{R}^d is a tiling, which we shall call a *cellular tiling*. There are some natural systems of internal symmetries defined over $\mathcal{F}(\mathcal{T}) := \mathcal{T}$.

Firstly, set the radius poset as $(\mathbb{R}_{>0}, \leq)$, with the usual linear ordering. If we wish to compare patches of tiles only up to translation, then we define \mathfrak{T}^1 as follows. For $a \in \mathcal{T}$ and $r \in \mathbb{R}_{>0}$, denote by $P(a, r)$ the ‘patch’ of cells within radius r of a . This may be defined in a variety of essentially equivalent ways (up to tail-equivalence, see Definition 3.4). For example, one may set

$$P(a, r) := \overline{\{a' \in \mathcal{T} \mid d(a, a') \leq r\}}$$

where the distance $d(a, a')$ is the infimum of (Euclidean) distances between points of a and a' . Then set $\Phi \in (\mathfrak{T}^1)_{a,b}^r$ if and only if Φ is a cellular isomorphism between the star complexes of a and b induced by a translation taking $P(a, r)$ to $P(b, r)$ and preserving labels of cells, in case we have labelled the cells of our tiling.

One may instead wish to compare patches using orientation preserving isometries (which we shall call *rigid motions*). In this case, we define the SIS \mathfrak{T}^0 by setting $\Phi \in (\mathfrak{T}^0)_{a,b}^r$ if and only if Φ is a cellular isomorphism between the star complexes of a and b induced by a rigid motion taking $P(a, r)$ to $P(b, r)$.

For the purpose of studying topological invariants of these tilings, the restriction to cellular tilings is not a severe one. For any Euclidean tiling T (whose tiles have bounded in and out-radii), by making a selection of puncture for each cell one may define a Delone set P_T , and to that an associated Voronoi tiling $\mathcal{V}(P_T)$. The tiling $\mathcal{V}(P_T)$ defines a regular CW decomposition of \mathbb{R}^d of convex polytopal cells and, if one chooses the punctures carefully (or perhaps endows them with a colouring), the tilings $\mathcal{V}(P_T)$ and T are equivalent in a very rigid sense (they are ‘MLD’, see [34, Chapter 1, §3]).

2.1.2. *Combinatorial Tilings.* One may mimic the constructions above for the case that \mathcal{T} is a general CW poset, not necessarily coming from a cellular tiling of Euclidean space. One would usually think of a combinatorial tiling as defining a *pure*

d -dimensional complex, in which case it makes sense to think of the d -cells as the tiles, although this is not actually necessary here. We see that we need the notions of:

- (1) Patches of size λ .
- (2) Labelling of cells.
- (3) ‘Allowed symmetries’ of patches (c.f., Euclidean translations/rigid motions).

For item one, we may define a combinatorial distance between cells. Inductively define

$$\begin{aligned} \text{St}^0(a) &:= \overline{\text{St}(a)}, \\ \text{St}^n(a) &:= \overline{\{a'' \in \mathcal{T} \mid a'' \in \text{St}(a') \text{ for some } a' \in \text{St}^{n-1}(a)\}}. \end{aligned}$$

So for $n \in \mathbb{N}_0$ and $a \in \mathcal{T}$ one may define the n -patch at a to be $P(a, n) := \text{St}^n(a)$. For example, for a cellular tiling of \mathbb{R}^d and d -cell a of it, we have that $P(a, n)$ corresponds to what is often named the “ n -collared patch of a ” (c.f., the construction of the Gähler approximants; see [34, Chapter 2, §4]).

One may deal with items two and three together as follows. We let L be a groupoid, with objects the cells of \mathcal{T} and with morphisms $L_{a,b}$ between a and b cellular isomorphisms between cell posets \bar{a} and \bar{b} . So we demand that L contains identity morphisms, and is closed under composition and inversion. This allows one to label cells and to rule out certain cellular isomorphisms as non-allowed symmetries. For example, if \mathcal{T} is pure of dimension d and is oriented then one could allow for only orientation preserving morphisms between d -cells. We then define \mathfrak{T}^L over \mathcal{T} and radius poset (\mathbb{N}_0, \leq) by setting $\Phi \in (\mathfrak{T}^L)_{a,b}^n$ if and only if Φ is a cellular isomorphism between the star complexes of a and b induced by a cellular isomorphism between $P(a, n)$ and $P(b, n)$, sending a to b , whose restrictions to each cell of $P(a, n)$ are elements of the allowed morphisms of L .

2.1.3. Barycentric Subdivisions. Recall from Definition 1.4 that one may associate to a CW poset \mathcal{F} its barycentric subdivision \mathcal{F}_Δ . Given a system of internal symmetries \mathfrak{T} over a CW poset \mathcal{T} and radius poset Λ , we may define another, its *barycentric subdivision* \mathfrak{T}_Δ , defined over the CW poset \mathcal{T}_Δ and with radius poset Λ . To do so, firstly recall that given $\Phi \in \mathfrak{T}_{a,b}^\lambda$ we may define a cellular isomorphism Φ_Δ between the barycentric subdivisions of the star complexes of a and b (see §1.2). Given a simplex $s = \{a_0, \dots, a_n\}$ of \mathcal{F}_Δ we have that

$$\overline{\text{St}(s)} = \{s'' \in \mathcal{T}_\Delta \mid \exists s' : s \subseteq s' \supseteq s''\} \subseteq (\overline{\text{St}(a_n)})_\Delta.$$

So we may define \mathfrak{T}_Δ by setting $\Phi \in (\mathfrak{T}_\Delta)_{s,t}^\lambda$ if and only if there exists some $\tilde{\Phi} \in \mathfrak{T}^\lambda$ between the star complexes of the top entries of s and t for which $\tilde{\Phi}_\Delta$ restricts to Φ .

Proposition 2.2. Let \mathfrak{T} be an SIS. Then \mathfrak{T}_Δ is also an SIS.

Proof. The first three groupoid axioms for an SIS follow easily from the fact that $\text{Id}_{\overline{\text{St}(c)}_\Delta} = (\text{Id}_{\overline{\text{St}(c)}})_\Delta$, $(\Phi_2)_\Delta \circ (\Phi_1)_\Delta = (\Phi_2 \circ \Phi_1)_\Delta$ and $(\Phi_\Delta)^{-1} = (\Phi^{-1})_\Delta$. Similarly, the fourth axiom (Inc) follows trivially.

For the fifth axiom (Res) of restriction, let $\lambda \in \Lambda$ and $s \leq t$ be simplexes of \mathcal{T}_Δ whose vertices of maximal dimension are $a, b \in \mathcal{T}$, respectively. Set λ_{res} as in axiom (Res) for \mathfrak{T} . Suppose that $\Phi \in (\mathfrak{T}_\Delta)_{s,-}^{\lambda_{\text{res}}}$. Then Φ is a restriction to the star complex of s of $\tilde{\Phi}_\Delta$ for some $\tilde{\Phi} \in \mathfrak{T}_{a,-}^{\lambda_{\text{res}}}$. So, by (Res), the restriction Φ' of $\tilde{\Phi}$ to the star complex of b is an

element of $\mathfrak{T}_{b,-}^\lambda$. The restriction of Φ'_Δ to the star complex of t is an element of $(\mathfrak{T}_\Delta)_{t,-}^\lambda$ which is a restriction of Φ , establishing axiom (Res) for \mathfrak{T}_Δ . Axiom (CoRes) is proved analogously. \square

3. PE (Co)HOMOLOGY AND PE POINCARÉ DUALITY

3.1. PE (Co)homology. Given a system of internal symmetries, one may consider the Borel–Moore chains, or cochains, which assign oriented coefficients to cells in a way which depends only on the equivalence classes of the cells to some sufficiently large radius:

Definition 3.1. Let \mathfrak{T} be an SIS over the CW poset \mathcal{T} and radius poset Λ . We shall say that a Borel–Moore chain $\sigma \in C_n(\mathcal{T})$ is *pattern-equivariant (PE) to radius $\lambda \in \Lambda$* if for any $\Phi \in \mathfrak{T}_{a,b}^\lambda$ between n -cells a and b we have that $\sigma(\omega_a) = \sigma(\Phi_*(\omega_a))$, where ω_a is some orientation on a . Similarly, we say that a cochain $\psi \in C^n(\mathcal{T})$ is *pattern-equivariant to radius λ* if, for any $\Phi \in \mathfrak{T}_{a,b}^\lambda$ we have that $\psi(\omega_a) = \psi(\Phi_*(\omega_a))$.

Proposition 3.2. The (co)boundary of a PE (co)chain is PE.

Proof. Let σ be a Borel–Moore n -chain which is PE to radius λ . We must check that there exists some λ' for which, for any two $(n-1)$ -cells a, b and $\Phi \in \mathfrak{T}_{a,b}^{\lambda'}$, we have that $\partial(\sigma)(\omega_a) = \partial(\sigma)(\Phi_*(\omega_a))$. In fact, we claim that $\lambda' = \lambda_{\text{res}}$ as in axiom (Res) of Definition 2.1 for \mathfrak{T} will do.

Indeed, given $\Phi \in \mathfrak{T}_{a,b}^{\lambda_{\text{res}}}$, the restriction of Φ to each star complex of n -cell containing a is an element of \mathfrak{T}^λ . It follows, by the fact that σ is PE to radius λ , that the cellular isomorphism Φ not only maps the star complex at a to the star complex at b , but it also maps $\sigma|_{\text{St}(a)}$ to $\sigma|_{\text{St}(b)}$. Since $\partial(\sigma)(\omega_a)$ only depends on the restriction of σ to $\text{St}(a)$ (and similarly for b), we see that $\partial(\sigma)(\omega_a) = \partial(\sigma)(\Phi_*(\omega_a))$, so $\partial(\sigma)$ is PE to radius λ_{res} . The proof that the coboundary of a PE cochain is PE is analogous; one implements axiom (CoRes) instead. \square

It follows from the above that the chain complex $C_\bullet^{\text{BM}}(\mathcal{T})$ restricts to a sub-chain complex $C_\bullet(\mathfrak{T})$ of PE chains. So we may define the *pattern-equivariant homology* of an SIS \mathfrak{T} to be $H_\bullet(\mathfrak{T}) := H(C_\bullet(\mathfrak{T}))$. We similarly have the cochain complex $C^\bullet(\mathfrak{T})$ of PE cochains and associated *pattern-equivariant cohomology* $H^\bullet(\mathfrak{T}) := H(C^\bullet(\mathfrak{T}))$.

Example 3.3. Consider the simple example of the periodic tiling T of \mathbb{R}^2 by unit squares, each with corners lying on the integer lattice \mathbb{Z}^2 . This defines a CW poset \mathcal{T} and the SIS \mathfrak{T}^1 based upon comparison of patches via translation (see §2.1.1). There are naturally four types of cells of \mathcal{T} : vertices, horizontal edges, vertical edges and faces. The groupoid $(\mathfrak{T}^1)^r$ has the same description for each radius $r \in \mathbb{R}_{>0}$: we have precisely one morphism (induced by translation) in each $(\mathfrak{T}^1)_{a,b}^\lambda$ whenever a, b are of the same type, and $(\mathfrak{T}^1)_{a,b}^\lambda = \emptyset$ otherwise. An n -(co)chain being PE simply means that all n -cells of the same type have the same (oriented) coefficient. So $C_i(\mathfrak{T}) \cong \mathbb{Z}, \mathbb{Z}^2$ and \mathbb{Z} for $i = 0, 1, 2$, respectively (and similarly for PE cohomology). The (co)boundary maps are easily calculated to be trivial, and so the PE (co)homology groups have the same description. The PE cohomology is isomorphic to the cohomology of the tiling space of

T (see §3.2), which is the 2-torus \mathbb{T}^2 , and the PE homology is Poincaré dual to the PE cohomology; see Theorem 3.11.

Definition 3.4. Let \mathfrak{T}_1 and \mathfrak{T}_2 be two SIS's, defined over the same CW poset and radius posets Λ_1 and Λ_2 , respectively. If for all $\lambda_2 \in \Lambda_2$ there exists some $\lambda_1 \in \Lambda_1$ for which $\mathfrak{T}_1^{\lambda_1} \subseteq \mathfrak{T}_2^{\lambda_2}$, and vice versa, we say that \mathfrak{T}_1 and \mathfrak{T}_2 are *tail-equivalent*.

Clearly tail-equivalence defines an equivalence relation on SIS's defined over a fixed CW poset \mathcal{T} and, for \mathfrak{T}_1 and \mathfrak{T}_2 tail-equivalent, a (co)-chain is PE in \mathfrak{T}_1 if and only if it is PE in \mathfrak{T}_2 .

Example 3.5. Let T be a cellular tiling of \mathbb{R}^d whose tiles have bounded in and out-radii. Then whether one defines \mathfrak{T}^1 or \mathfrak{T}^0 in terms of patches of radius $r \in \mathbb{R}_{>0}$ or in terms of n -collared patches for $n \in \mathbb{N}_{>0}$ does not matter up to tail-equivalence.

The groups $\mathfrak{T}_{a,a}^\lambda$ of \mathfrak{T}^λ shall be called the *local isotropy groups*. If $\mathfrak{T}_{a,a}^\lambda = \{\text{Id}_{\overline{\text{St}(a)}}\}$ for all $a \in \mathcal{T}$ and sufficiently large λ , then we shall say that \mathfrak{T} has *trivial local isotropy*.

Denote by $\tilde{\mathfrak{T}}^\lambda$ the collection of restrictions of \mathfrak{T}^λ to source cells, so that $\tilde{\Phi} \in \tilde{\mathfrak{T}}^\lambda$ if and only if $\tilde{\Phi} = \Phi|_{\bar{a}}$ for some $\Phi \in \mathfrak{T}_{a,-}^\lambda$ (we shall always implicitly ‘corestrict’ cellular isomorphisms to the appropriate range). We write $\tilde{\Phi} \in \tilde{\mathfrak{T}}_{a,b}^\lambda$ to specify that $\tilde{\Phi}$ is the restriction of some $\Phi \in \mathfrak{T}_{a,b}^\lambda$ between \bar{a} and \bar{b} . Then each $\tilde{\mathfrak{T}}^\lambda$ is a groupoid. We say that \mathfrak{T} has *trivial cell isotropy* if $\tilde{\mathfrak{T}}_{a,a}^\lambda = \{\text{Id}_{\bar{a}}\}$ for all $a \in \mathcal{T}$ and sufficiently large λ . Note that the isotropy groups of cells $\tilde{\mathfrak{T}}_{a,a}^\lambda$ are quotient groups of the local isotropy groups $\mathfrak{T}_{a,a}^\lambda$.

A commutative ring G has *division by n* for $n \in \mathbb{N}$ if $n \cdot 1$ is invertible in G .

Proposition 3.6. Let \mathfrak{T} be an SIS, and fix a coefficient ring G . Suppose that, for sufficiently large $\lambda \in \Lambda$, we have that G has division by the order of the cell isotropy group $\#(\tilde{\mathfrak{T}}_{a,a}^\lambda)$ for all cells $a \in \mathcal{T}$. Then we have quasi-isomorphisms:

$$\begin{aligned} \iota_\bullet &: C_\bullet(\mathfrak{T}; G) \rightarrow C_\bullet(\mathfrak{T}_\Delta; G) \\ \iota^\bullet &: C^\bullet(\mathfrak{T}_\Delta; G) \rightarrow C^\bullet(\mathfrak{T}; G) \end{aligned}$$

Proof. Mimicking the proof Proposition 1.11, upon restricting to PE chains we need to show that

- (1) $H_n(\mathfrak{T}_\Delta^k, \mathfrak{T}_\Delta^{k-1}) = 0$ for $n \neq k$
- (2) We have a canonical isomorphism of chain complexes $C_\bullet(\mathfrak{T}) \cong H_\bullet(\mathfrak{T}_\Delta^\bullet, \mathfrak{T}_\Delta^{\bullet-1})$

(the notation of which shall be explained below). For the proof of the quasi-isomorphism in homology, we shall use a simple ‘averaging trick’, implementing the divisibility of G , to show 1. One may obtain part 2 in homology ‘for free’. In the cohomology case, the situation is reversed, and one essentially needs the divisibility of G for part 2.

Let $C_n(\mathfrak{T}_\Delta^k, \mathfrak{T}_\Delta^{k-1})$ be the group of equivalence classes of PE chains of \mathfrak{T}_Δ supported on \mathcal{T}_Δ^k , where we identify two such chains if they agree away from \mathcal{T}_Δ^{k-1} . The usual boundary maps induce boundary maps between the relative groups (for fixed k), so one may define the homology groups $H_n(\mathfrak{T}_\Delta^k, \mathfrak{T}_\Delta^{k-1})$; we wish to show that these groups are trivial for $n \neq k$.

So let $n < k$ and $\sigma \in C_n(\mathfrak{T}_\Delta^k)$ be a PE simplicial n -chain supported on \mathcal{T}_Δ^k for which $\partial(\sigma)$ is supported on the \mathcal{T}_Δ^{k-1} . The chain σ is PE to some radius λ , which we may assume to be sufficiently large relative to the condition of the proposition. We know that there exists some Borel–Moore $(n+1)$ -chain τ for which $\sigma + \partial(\tau)$ is supported on the $(k-1)$ -skeleton, we just need to show that τ may be chosen to be PE.

Say that k -cells $a, b \in \mathcal{T}$ are equivalent to radius λ if $\mathfrak{T}_{a,b}^\lambda \neq \emptyset$, which defines equivalence relations on the k -cells by the groupoid axioms for \mathfrak{T} . Choose an equivalence class $[a]$ of cells and representative $a \in [a]$ for radius λ . The restriction of σ to a_Δ° is homologous to a chain supported on ∂a_Δ via some $(n+1)$ -chain τ_a supported on a_Δ° . That is, we have a chain τ_a supported on a_Δ° for which

$$(\sigma \upharpoonright a_\Delta^\circ) + \partial(\tau_a) = \sigma_{\partial(a)},$$

for some chain $\sigma_{\partial(a)}$ supported on $\partial a_\Delta \subseteq \mathcal{T}_\Delta^{k-1}$. Define the PE $(n+1)$ -chain $\tau_{[a]}$ by setting

$$\tau_{[a]} := \sum_{\Phi \in \tilde{\mathfrak{T}}_{a,-}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \cdot \Phi_*(\tau_a),$$

where the sum is taken over all morphisms of $\tilde{\mathfrak{T}}^\lambda$ with source a . To show that the above is PE, note firstly that $\tau_{[a]}$ is supported on the cells which are equivalent to a to radius λ . So it suffices to show that the morphisms $\Phi \in \mathfrak{T}_{b,c}^\lambda$, where $[a] = [b] = [c]$, sends $\tau_{[a]}$ restricted to b to $\tau_{[a]}$ restricted to c . Indeed, we have that:

$$\begin{aligned} \Phi_*(\tau_{[a]} \upharpoonright b_\Delta^\circ) &= \Phi_* \left(\sum_{\Phi' \in \tilde{\mathfrak{T}}_{a,-}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \cdot \Phi'_*(\tau_a) \right) \upharpoonright b_\Delta^\circ = \\ \sum_{\Phi' \in \tilde{\mathfrak{T}}_{a,b}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \Phi_* \circ \Phi'_*(\tau_a) &= \sum_{\Phi' \in \tilde{\mathfrak{T}}_{a,c}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \Phi'_*(\tau_a) = \tau_{[a]} \upharpoonright c_\Delta^\circ \end{aligned}$$

It follows that $\tau_{[a]}$ is PE in \mathfrak{T}_Δ . Since σ is PE to radius λ , we have that $\Phi_*(\sigma \upharpoonright a_\Delta^\circ) = \sigma \upharpoonright b_\Delta^\circ$ for all k -cells b and $\Phi \in \mathfrak{T}_{a,b}^\lambda$. Hence, for any cell b with $[a] = [b]$, we have that:

$$\begin{aligned} (\sigma \upharpoonright b_\Delta^\circ) + \partial(\tau_{[a]} \upharpoonright b_\Delta^\circ) &= (\sigma \upharpoonright b_\Delta^\circ) + \sum_{\Phi \in \tilde{\mathfrak{T}}_{a,b}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \Phi_*(\partial(\tau_a)) = \\ (\sigma \upharpoonright b_\Delta^\circ) + \sum_{\Phi \in \tilde{\mathfrak{T}}_{a,b}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} ((-\sigma \upharpoonright b_\Delta^\circ) + \Phi_*(\sigma_{\partial(a)})) &= \sum_{\Phi \in \tilde{\mathfrak{T}}_{a,b}^\lambda} \#(\tilde{\mathfrak{T}}_{a,a}^\lambda)^{-1} \cdot \Phi_*(\sigma_{\partial(a)}) \end{aligned}$$

This final term is a chain supported on the boundary of b .

Define the chain τ as the sum of chains $\tau_{[a]}$ taken over all equivalence classes of cells. Then it follows from the above that τ is PE and that $\sigma + \partial(\tau)$ is a chain supported on \mathcal{T}_Δ^{k-1} , as desired.

The chain isomorphism $C_\bullet(\mathfrak{T}) \cong H_\bullet(\mathfrak{T}_\Delta^\bullet, \mathfrak{T}_\Delta^{\bullet-1})$ of part 2 above is defined by the chain map $(-)_\Delta$ which canonically associates to a cellular chain its corresponding relative simplicial cycle (see the proof of Proposition 1.11). It is easily checked that $(-)_\Delta$ restricts to an isomorphism between the PE chain complexes.

We shall omit the proof in the cohomology case, which is similar. We note, however, that for part 2 one needs to implement the divisibility of G . Indeed, in this case, an

n -cochain ψ_Δ is defined for an n -cochain ψ with $\psi(\omega_c) = g$ by making a choice of simplicial n -cochain for each closed n -cell c_Δ for which the oriented coefficients *sum* to g . For non-divisible coefficient ring G , it may not be possible to define such a cochain to be PE, even if ψ is, if the cells have non-trivial isotropy in \mathfrak{T} . \square

For an SIS \mathfrak{T} we have that \mathfrak{T}_Δ has trivial cell isotropy (indeed, any element of $(\tilde{\mathfrak{T}}_\Delta)_{s,s}^\lambda$ must preserve the canonical total ordering of the vertices of the simplex s). So by the above proposition, the PE (co)homology is stable under barycentric subdivision after passing to the first barycentric subdivision. We view the PE (co)homology of \mathfrak{T} in the case of having non-trivial cell isotropy as being potentially ‘incorrect’. When \mathfrak{T} has trivial cell isotropy then the PE cohomology of \mathfrak{T} will correspond to the Čech cohomology of an associated tiling space (see the following subsection). The PE homology will only necessarily correspond to a singular version in general when \mathfrak{T} has trivial cell isotropy [38].

3.2. Tiling Spaces. For an FLC tiling T of \mathbb{R}^d , there is an associated topological space Ω^1 , called the *translational hull* or *tiling space* of T [34]. It may be defined by taking the collection $T + \mathbb{R}^d$ of translates of the tiling and then taking the completion of this collection with respect to the tiling metric, which deems two tilings to be ‘close’ if they are identical to a ‘large’ radius about the origin, up to a ‘small’ perturbation. Taking the completion is the same as considering all tilings ‘locally indistinguishable’ from T , tilings for which any finite sub-patch appears as a translate of a sub-patch of T .

As briefly justified in §2.1.1, for the purpose of studying a tiling T through its associated tiling space Ω^1 , it does no harm to restrict to the case that T is cellular. Then the tiling space Ω^1 is homeomorphic to an inverse limit of CW complexes, called *approximants*, which may be constructed by glueing together “collared tiles” (in the Gähler construction, see [34, Chapter 2, §4]) or by “collaring points” (in the Barge–Diamond–Hunton–Sadun construction, see [4, §3]). In direct analogy to the former approach, for any SIS there is an associated inverse limit of CW complexes. Suppose that \mathfrak{T} is an SIS, and that $\lambda_{\widehat{\text{res}}}$ may be set to be λ in axiom (Res) of Definition 2.1. Note that we may always choose a tail-equivalent SIS for which this is the case. In addition, suppose that the isotropy of cells is trivial, which will at least always be the case after passing to the barycentric subdivision. Recall from the previous subsection that $\tilde{\mathfrak{T}}^\lambda$ denotes the collection of morphisms of \mathfrak{T}^λ restricted to source cells. Then each $\tilde{\mathfrak{T}}^\lambda$ satisfies:

- (1) For each $a \in \mathcal{T}$, the identity morphism $\text{Id}_a \in \tilde{\mathfrak{T}}^\lambda$.
- (2) For each $\Phi \in \tilde{\mathfrak{T}}^\lambda$, we have that $\Phi^{-1} \in \tilde{\mathfrak{T}}^\lambda$.
- (3) For $\Phi_1 \in \tilde{\mathfrak{T}}_{a,b}^\lambda$ and $\Phi_2 \in \tilde{\mathfrak{T}}_{b,c}^\lambda$, we have that $\Phi_2 \circ \Phi_1 \in \tilde{\mathfrak{T}}_{a,c}^\lambda$.
- (4) If $\Phi \in \tilde{\mathfrak{T}}_{c,c}^\lambda$, then Φ is the identity morphism.
- (5) If $\Phi \in \tilde{\mathfrak{T}}_{b,-}^\lambda$ and b is a coface of a , then the restriction of Φ to a is an element of $\tilde{\mathfrak{T}}^\lambda$.

By passing to a geometric realisation of the above system of morphisms, the above axioms precisely specify a *family of identifications* of the cells \mathcal{T} , see [12, Chapter III]. One may consider a quotient of $|\mathcal{T}|$ by identifying the cells via the morphisms of

\mathfrak{T}^λ . The quotient inherits a CW decomposition, although not necessarily a regular one (without further barycentric subdivision).

The quotient spaces $K_\lambda = |\mathcal{T}|/\mathfrak{T}^\lambda$ shall be called *approximants*. Denote the quotient map from $|\mathcal{T}|$ to such an approximant by π_λ . For $\lambda \leq \mu$, since $\mathfrak{T}^\lambda \supseteq \mathfrak{T}^\mu$, we have cellular quotient maps $\pi_{\lambda,\mu}: K_\mu \rightarrow K_\lambda$ given by making any extra identifications of $\tilde{\mathfrak{T}}^\lambda$ that may not have been identifications for $\tilde{\mathfrak{T}}^\mu$. Then \mathfrak{T} defines an inverse system $(K_\lambda, \pi_{\lambda,\mu})$ indexed over the directed set Λ , whose inverse limit will be denoted by $\Omega(\mathfrak{T})$ and is called the *tiling space* of \mathfrak{T} .

Proposition 3.7. Let \mathfrak{T} be an SIS satisfying conditions 4 and 5 above. If, in addition, there are only a finite number of equivalence classes of cells of $\tilde{\mathfrak{T}}^\lambda$ for each $\lambda \in \Lambda$, then there exists an isomorphism $\check{H}^\bullet(\Omega(\mathfrak{T})) \cong H^\bullet(\mathfrak{T})$.

Proof. The proof is essentially the same as that of [33, Theorem 4]. Under the above hypotheses, $\Omega(\mathfrak{T})$ is an inverse limit over Λ of compact Hausdorff spaces, and so is itself compact (and Hausdorff). From the compactness of $\Omega(\mathfrak{T})$, it is easy to show that $\check{H}^\bullet(\Omega(\mathfrak{T})) \cong \varinjlim (\check{H}^\bullet(K_\lambda, \pi_{\lambda,\mu}^*)$; we note that compactness cannot be dropped here, there are inverse limits of non-compact CW complexes for which this does not hold. Čech cohomology is naturally isomorphic to singular cohomology on the subcategory of topological spaces which are homotopy equivalent to CW complexes. So we have that

$$\check{H}^\bullet(\Omega(\mathfrak{T})) = \check{H}^\bullet(\varprojlim (K_\lambda, \pi_{\lambda,\mu})) \cong \varinjlim (\check{H}^\bullet(K_\lambda, \pi_{\lambda,\mu}^*) \cong \varinjlim (H^\bullet(K_\lambda, \pi_{\lambda,\mu}^*),$$

where $H^\bullet(K_\lambda)$ is the cellular cohomology of K_λ . Taking cohomology commutes with taking the direct limit, and so the above is isomorphic to the cohomology of the direct limit cochain complex $\varinjlim (C^\bullet(K_\lambda), \pi_{\lambda,\mu}^*)$. A cellular cochain $\psi \in C^n(\mathcal{T})$ is PE if and only if it is the pullback $\pi_\lambda^*(\tilde{\psi})$ of a cellular cochain $\tilde{\psi} \in C^\bullet(K_\lambda)$ for some $\lambda \in \Lambda$. We see that the maps π_λ^* induce a cochain isomorphism from the cochain complex $\varinjlim (C^\bullet(K_\lambda), \pi_{\lambda,\mu}^*)$ to $C^\bullet(\mathfrak{T})$ and so $\check{H}^\bullet(\Omega(\mathfrak{T})) \cong H^\bullet(\mathfrak{T})$. \square

Example 3.8. Let T be an FLC cellular tiling of \mathbb{R}^d and consider the SIS \mathfrak{T}^1 based upon comparison of patches via translations (see §2.1.1). Finite local complexity (FLC) here means that there are only finitely many patches of any given radius r up to translational equivalence. The tiling space $\Omega(\mathfrak{T}^1)$ is an inverse limit of approximants constructed by identifying cells when they agree to a certain radius in the tiling up to translation. This is precisely the translational hull of T , which in [4] is denoted by Ω^1 . So we have that $\check{H}^\bullet(\Omega^1) \cong H^\bullet(\mathfrak{T}^1)$, as shown in [33].

Similarly, one may define the SIS \mathfrak{T}^0 based upon comparison of patches via rigid motions. Suppose that T is *rotationally FLC*, which means that for each $r > 0$ there are only a finite number of patches of size r up to rigid motion (Radin's pinwheel tilings are the standard examples of tilings which are rotationally FLC, but are not FLC with respect to translations). The space $\Omega(\mathfrak{T}^0)$ is an inverse limit of approximants built by identifying cells whenever they agree to a certain radius in the tiling up to rigid motion. We require here that the cells have trivial isotropy in \mathfrak{T}^0 , which is always the case after passing to the barycentric subdivision. The space $\Omega(\mathfrak{T}^0)$ corresponds to the space $\Omega^0 \cong \Omega^{\text{rot}}/\text{SO}(d)$ of [4], where Ω^{rot} is the full Euclidean hull of T , which is acted

upon by $\mathrm{SO}(d)$ by rotations. So again, the Čech cohomology of this space may be visualised using PE cochains: $\check{H}^\bullet(\Omega^0) \cong H^\bullet(\mathfrak{T}^0)$, c.f., [31, 34].

Example 3.9. Let \mathcal{S}^d be the CW poset corresponding to the periodic tiling of \mathbb{R}^d by unit hypercubes with corners lying on the integer lattice \mathbb{Z}^d . We define an SIS \mathfrak{D}_2^d over \mathcal{S}^d and radius poset (\mathbb{N}_0, \leq) as follows. Let $\Phi \in (\mathfrak{D}_2^d)_{a,b}^n$ if and only if Φ is a cellular isomorphism between the star complexes of a and b induced by translation by a vector of $2^n \mathbb{Z}^d$.

It is easy to see that each approximant K_n is homeomorphic to the d -torus \mathbb{T}^d and that the map $\pi_{i,j}$ for $i \leq j$ corresponds to the self-map of \mathbb{T}^d induced by the $\times 2^{j-i}$ map on \mathbb{R}^d . So the tiling space $\Omega(\mathfrak{D}_2^d)$ is the d -dimensional dyadic solenoid $\mathbb{D}_2^d := \varprojlim (\mathbb{T}^d, \times 2)$. A cochain $\psi \in C^n(\mathcal{S}^d)$ is PE if and only if there exists some $n \in \mathbb{N}_0$ for which ψ is invariant under the action of translation of $2^n \mathbb{Z}^d$. The PE cohomology is isomorphic to the Čech cohomology of the tiling space $\check{H}^\bullet(\Omega(\mathfrak{D}_2^d)) \cong H^\bullet(\mathbb{D}_2^d)$. This example may be thought of as a hierarchical tiling, as described in §4.5.5, where the cells ‘know’ of their location in the hierarchy despite this information not being determined geometrically by the underlying tiling. The ‘supertilings’ T_n in the hierarchy are periodic tilings of hypercubes of side-length 2^n , which are the ‘supertiles’ of the non-recognisable substitution of a unit hypercube into 2^d copies of half the side-length. Alternatively, it may be realised as a tiling whose tiles are labelled by elements of a compact metric space, see [29, Example 3].

3.3. PE Poincaré Duality. As we have seen, the PE cohomology of an SIS \mathfrak{T} corresponds to the Čech cohomology of an associated inverse limit space $\Omega(\mathfrak{T})$, which in the main cases of interest (for example, for the translational hull of an FLC tiling) corresponds to a naturally defined moduli space of tilings. We now wish to show that the PE homology, in certain cases, is related to the PE cohomology through Poincaré duality.

Definition 3.10. Let \mathfrak{T} be an SIS over a pure CW poset \mathcal{T} . We define the chain complex $C_\bullet(\widehat{\mathfrak{T}}; G)$ to be the sub-chain complex of $C_\bullet(\mathfrak{T}_\Delta; G)$ which in degree n consists of PE simplicial n -chains supported on the dual n -skeleton $\widehat{\mathcal{T}}_\Delta^n$ and with boundaries supported on the dual $(n-1)$ -skeleton $\widehat{\mathcal{T}}_\Delta^{n-1}$.

Theorem 3.11 (PE Poincaré Duality). Let \mathfrak{T} be an SIS over a CW poset \mathcal{T} for a homology d -manifold $|\mathcal{T}|$. Set a coefficient ring G and assume that there exists a PE fundamental class $\Gamma \in C_d(\mathfrak{T}; G)$ (that is, Γ is a Borel–Moore fundamental class of $C_d^{\mathrm{BM}}(\mathcal{T}; G)$ and the morphisms of \mathfrak{T}^λ between d -cells are orientation preserving for sufficiently large λ). Then the cap product with the fundamental class induces a cochain isomorphism

$$- \cap \Gamma: C^\bullet(\mathfrak{T}; G) \rightarrow C_{d-\bullet}(\widehat{\mathfrak{T}}; G).$$

Furthermore, if for sufficiently large λ we have that G has division by the order of local isotropy $\#(\mathfrak{T}_{a,a}^\lambda)$ for all cells $a \in \mathcal{T}$, then there exists a quasi-isomorphism

$$\iota_\bullet: C_\bullet(\widehat{\mathfrak{T}}; G) \rightarrow C_\bullet(\mathfrak{T}_\Delta; G).$$

Hence, in this case, we have PE Poincaré duality $H^\bullet(\mathfrak{T}; G) \cong H_{d-\bullet}(\mathfrak{T}; G)$.

Proof. Each $\Phi \in \mathfrak{T}_{a,b}^\lambda$ induces a cellular isomorphism $\widehat{\Phi}$ between the dual cells \widehat{a}_Δ and \widehat{b}_Δ by setting

$$\widehat{\Phi}(\{a_0, \dots, a_n\}) := \{\Phi(a_0), \dots, \Phi(a_n)\}$$

for a simplex $\{a_0, \dots, a_n\} \in \widehat{a}_\Delta$. It is simple to check that the above is well-defined and that:

- (1) $\text{Id}_{\widehat{a}_\Delta} = \widehat{\text{Id}_{\text{St}(a)}}$
- (2) $\widehat{\Phi}_2 \circ \widehat{\Phi}_1 = \widehat{\Phi_2 \circ \Phi_1}$
- (3) $(\widehat{\Phi})^{-1} = \widehat{\Phi^{-1}}$

It follows from the above and the groupoid axioms of \mathfrak{T} that we may define the groupoids $\widehat{\mathfrak{T}}^\lambda$ which have as objects the dual cells of \mathcal{T} and as morphisms the duals $\widehat{\Phi}$ of the morphisms of \mathfrak{T}^λ , analogously to the groupoids $\widetilde{\mathfrak{T}}^\lambda$.

The relative pair $(\widehat{a}_\Delta, \partial\widehat{a}_\Delta)$ of the dual of an n -cell modulo the dual boundary has an orientation, a simplicial $(d-n)$ -chain with coefficients ± 1 supported on \widehat{a}_Δ and boundary supported on $\partial\widehat{a}_\Delta$. A chain $\sigma \in C_n^{\text{BM}}(\widehat{\mathcal{T}})$ is an element of $C_n(\widehat{\mathfrak{T}})$ if and only if there exists some λ for which, for any $\widehat{\Phi} \in \widehat{\mathfrak{T}}_{\widehat{a}_\Delta, \widehat{b}_\Delta}^\lambda$, we have that $\widehat{\Phi}_*(\sigma|_{\widehat{a}_\Delta}) = \sigma|_{\widehat{b}_\Delta}$.

Recall that to explicitly define $\psi \cap \Gamma$ for $\psi \in C^n(\mathcal{T})$, one firstly defines the simplicial cochain $\psi_\Delta \in C^n(\mathcal{T}_\Delta^n, \mathcal{T}_\Delta^{n-1})$ and then take its cap product (with respect to the canonical partial order of the vertices of \mathcal{T}_Δ) with the simplicial cycle $\Gamma_\Delta \in C_d(\mathcal{T}_\Delta)$. Then we have that $\psi \in C^n(\mathcal{T})$ is PE if and only if $\psi \cap \Gamma$ is PE. Indeed, suppose that all of the elements of $\mathfrak{T}^{\lambda'}$ preserve the orientations of d -cells, which is true for sufficiently large λ' by the assumption of the theorem and axiom (Res) of Definition 2.1. Then for any $\Phi \in \mathfrak{T}_{a,b}^\lambda$ for n -cells a and b with $\lambda \geq \lambda'$, we have that

$$\Phi_*(\Gamma|_{\text{St}(a)}) = \Gamma|_{\text{St}(b)}.$$

Since Φ transports the restriction of the chain Γ at the star of a to its restriction at the star of b , as well as the canonical partial ordering of the vertices, the result quickly follows from the classical pairing of the cell orientation with the corresponding dual cell orientation induced by the cap product.

The proof of the second quasi-isomorphism is entirely analogous to the proof of Proposition 3.6, where we use $\widehat{\mathfrak{T}}$ in replacement of $\widetilde{\mathfrak{T}}$. To use the ‘averaging trick’ one needs that, for sufficiently large λ , the coefficient ring G has division by the order of the isotropy of dual cells $\#(\widehat{\mathfrak{T}}_{\widehat{a}_\Delta, \widehat{a}_\Delta}^\lambda)$. Note that each $\widehat{\mathfrak{T}}_{\widehat{a}_\Delta, \widehat{a}_\Delta}^\lambda$ is a quotient group of the local isotropy group $\mathfrak{T}_{a,a}^\lambda$. Hence, G eventually has division by the order of the isotropy of dual cells whenever it eventually has division by the order of the local isotropy of \mathfrak{T} . In this case, similarly, G eventually has division by the order of the isotropy of cells $\#(\widetilde{\mathfrak{T}}_{a,a}^\lambda)$ and so by Proposition 3.6 the PE homology is invariant under barycentric subdivision. In summation, we have the following quasi-isomorphisms

$$C^\bullet(\mathfrak{T}) \rightarrow C_{d-\bullet}(\widehat{\mathfrak{T}}) \rightarrow C_{d-\bullet}(\mathfrak{T}_\Delta) \leftarrow C_{d-\bullet}(\mathfrak{T})$$

and so we have PE Poincaré duality $H^\bullet(\mathfrak{T}) \cong H_{d-\bullet}(\mathfrak{T})$. \square

PE Poincaré duality allows one to visualise topological invariants of tiling spaces using geometric chains supported on the original tiling. Since PE Poincaré duality is

simply a restriction of usual Poincaré duality, it is also often possible to express the product structure on the cohomology groups as an intersection product in PE homology.

As we shall see in examples to follow, PE Poincaré duality often fails in the case that the SIS has local isotropy. We shall show in §3.4 that one may modify the definition of PE homology so as to regain duality, although we consider the extra torsion elements seen in the PE homology as invariants of potential interest, which we shall utilise in forthcoming work [37].

Example 3.12. Let T be an FLC regular cellular tiling of \mathbb{R}^d and associate to it the SIS \mathfrak{T}^1 (see §2.1.1). Clearly \mathfrak{T}^1 has trivial local isotropy, and so we have Poincaré duality $H^\bullet(\mathfrak{T}^1) \cong H_{d-\bullet}(\mathfrak{T}^1)$. The PE cohomology is isomorphic to the Čech cohomology $\check{H}^\bullet(\Omega(\mathfrak{T}^1))$ of the translational hull of T .

Now consider \mathfrak{T}^0 , which is defined by allowing comparison of patches using rigid motions instead of just translations. Assume that the cells have trivial isotropy in \mathfrak{T}^0 , which will at least always be the case by passing to a barycentric subdivision. In this case we still have that $H^\bullet(\mathfrak{T}^0)$ is isomorphic to the Čech cohomology of the tiling space $\check{H}^\bullet(\Omega(\mathfrak{T}^0))$. However, it is not necessarily true that we have Poincaré duality $H^\bullet(\mathfrak{T}^0) \cong H_{d-\bullet}(\mathfrak{T}^0)$. If there exists a rotationally invariant tiling of the tiling space, then for all $r > 0$ there will exist some cell $a \in \mathcal{T}$ with $\#(\mathfrak{T}_{a,a}^r) > 1$ so Theorem 3.11 does not apply. One still has Poincaré duality over divisible coefficients, however. Alternatively, one may restore duality by modifying the PE chain complex, see §3.4.

Example 3.13. Let T be a periodic tiling of \mathbb{R}^2 of equilateral triangles. For \mathfrak{T}^1 we have Poincaré duality $\check{H}^\bullet(\Omega(\mathfrak{T}^1)) \cong H^\bullet(\mathfrak{T}^1) \cong H_{d-\bullet}(\mathfrak{T}^1)$, since $\#(\mathfrak{T}_{a,a}^\lambda) = 1$ for all $a \in \mathcal{T}$. The space $\Omega(\mathfrak{T}^1)$ is the translational hull of T , which is the 2-torus \mathbb{T}^2 .

To calculate invariants for \mathfrak{T}^0 , because of cell isotropy we pass to the barycentric subdivision \mathfrak{T}_Δ^0 . In this case, we do not have Poincaré duality: $H_i(\mathfrak{T}_\Delta^0) \cong H^{2-i}(\mathfrak{T}_\Delta^0) \cong 0, \mathbb{Z}$ for $i = 1, 2$, respectively, but for $i = 0$ we have that $H_0(\mathfrak{T}_\Delta^0) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ whereas $H^2(\mathfrak{T}_\Delta^0) \cong \mathbb{Z}$. We have that $H^\bullet(\mathfrak{T}_\Delta^0) \cong \check{H}^\bullet(\Omega(\mathfrak{T}_\Delta^0))$, where $\Omega(\mathfrak{T}_\Delta^0) \cong S^2$ is the 2-sphere.

We do not have Poincaré duality between the PE cohomology and PE homology here, since cells have *local* isotropy, rotational symmetries at the 0-cells. Carrying out similar computations for a periodic square tiling S and its associated SIS \mathfrak{S}_Δ^0 , one computes that $H_0(\mathfrak{S}_\Delta^0) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. In particular, we see that the PE homology is not a topological invariant of the tiling space $\Omega(\mathfrak{S}_\Delta^0) \cong \Omega(\mathfrak{T}_\Delta^0) \cong S^2$. It is not hard to show that, over divisible coefficients G , the PE homology $H_\bullet(\mathfrak{T}^0; G)$ for a tiling T of \mathbb{R}^d corresponds to the subgroup of $H_\bullet(\mathfrak{T}^1; G)$ of elements represented by ‘rotationally invariant chains’ (c.f., [4, Theorem 7]). We see from this example that this is no longer necessarily true over non-divisible coefficients.

Remark 3.14. The name *pattern-equivariant* homology was chosen so as to be consistent with the common usage of the term in the field of aperiodic tilings. However, in the context of the above example it seems more logical, linguistically, for the pattern-equivariant cohomology groups to refer to the G -equivariant cohomology groups, where G is the symmetry group of the tiling acting on \mathbb{R}^2 by rigid motions. These groups will correspond to the group cohomology of G (since \mathbb{R}^2 is contractible), or the cohomology

of \mathbb{R}^2/G as an orbifold. Thus, the term *pattern-invariant* may be neater in this context. In fact, it should not be too difficult to define a cohomology theory associated to our SIS's \mathfrak{T} in general, which is to the PE cohomology (as defined here) as orbifold cohomology is to the coarse cohomology of its underlying space. One would expect the PE homology groups here to be relaying some extra information about such groups through the extra torsion elements that they possess.

We further remark that the term *homology* in PE homology is intended to reflect the geometric construction of these groups, rather than to refer to any functorial properties.

3.4. Modified PE Homology Groups. As we have seen, pattern-equivariant Poincaré duality can fail for SIS's in the presence of non-trivial local isotropy. We shall now describe how one may modify the definition of PE homology so as to regain duality with the PE cohomology.

To simplify discussion, we shall restrict to the case where \mathfrak{T} is defined over a 2-dimensional CW poset \mathcal{T} . We assume that $|\mathcal{T}|$ is an orientable 2-manifold and that the elements of \mathfrak{T}^λ preserve orientations of 2-cells. We shall set our coefficient group to be \mathbb{Z} throughout. Furthermore, we assume that the isotropy of cells is trivial which, with our previous assumptions, is equivalent to $\mathfrak{T}_{a,a}^\lambda = \{\text{Id}_{\overline{\text{St}(a)}}\}$ for sufficiently large λ and all $a \in \mathcal{T}$ with $\dim(a) > 0$ (so the non-trivial local isotropy is concentrated at the vertices).

We shall write $\sigma \in C_n^\dagger(\mathfrak{T})$ to mean that σ is PE with respect to \mathfrak{T} and, if $n = 0$, then in addition we require that there exists some λ for which, for each vertex $v \in \mathcal{T}$ with orientation ω_v , we have that $\sigma(\omega_v) = \#(\mathfrak{T}_{v,v}^\lambda) \cdot k$ for some $k \in \mathbb{Z}$. We claim that for $\sigma \in C_n^\dagger(\mathfrak{T})$ we have that $\partial(\sigma) \in C_{n-1}^\dagger(\mathfrak{T})$. We know that the boundary of a PE chain is PE, so we need only check that for $\sigma \in C_1(\mathfrak{T})$ there exists some λ for which $\sigma(\omega_v)$ is divisible by $\#(\mathfrak{T}_{v,v}^\lambda)$ for any vertex v . Indeed, suppose that σ has PE radius λ . Then $\sigma|_{\text{St}(v)}$ is invariant under $\#(\mathfrak{T}_{v,v}^{\lambda_{\text{res}}})$ -fold rotation and so $\partial(\sigma)(\omega_v)$ is some multiple of $\#(\mathfrak{T}_{v,v}^{\lambda_{\text{res}}})$. It follows that $C_\bullet^\dagger(\mathfrak{T})$ is a subcomplex of $C_\bullet(\mathfrak{T})$. Define the *modified PE homology groups* $H_\bullet^\dagger(\mathfrak{T}) := H(C_\bullet^\dagger(\mathfrak{T}))$. Then, with this modification, one has PE Poincaré duality:

Theorem 3.15. Let \mathfrak{T} be an SIS over the CW poset \mathcal{T} of an orientable homology d -manifold $|\mathcal{T}|$ which possesses a fundamental class $\Gamma \in C_d^{\text{BM}}(\mathcal{T})$. Suppose that the elements of $\mathfrak{T}_{a,a}^\lambda$ are orientation preserving on the 2-cells of \mathcal{T} and that \mathfrak{T} has trivial cell isotropy (but not necessarily trivial local isotropy). Then we have Poincaré duality $H^\bullet(\mathfrak{T}) \cong H_{d-\bullet}^\dagger(\mathfrak{T})$.

Proof. The proof is a simple modification of the proof for Theorem 3.11. One needs to show that there exist the following quasi-isomorphisms:

$$\begin{aligned} \iota_\bullet &: C_\bullet^\dagger(\mathfrak{T}) \rightarrow C_\bullet^\dagger(\mathfrak{T}_\Delta) \\ - \cap \Gamma &: C^\bullet(\mathfrak{T}) \rightarrow C_{d-\bullet}^\dagger(\widehat{\mathfrak{T}}) \\ \widehat{\iota}_\bullet &: C_\bullet^\dagger(\widehat{\mathfrak{T}}) \rightarrow C_\bullet^\dagger(\mathfrak{T}_\Delta) \end{aligned}$$

The first two follow with almost no alterations to the proofs of Proposition 3.6 and Theorem 3.11. Poincaré duality fails before the modification of the PE chain complex

because of the final chain map not necessarily being a quasi-isomorphism. It may not be true that $H_0(\widehat{\mathfrak{T}}^2, \widehat{\mathfrak{T}}^1) = 0$, since a 2-dimensional dual cell may have non-trivial isotropy to radius λ when the corresponding 0-cell of \mathcal{T} has non-trivial n -fold local isotropy. In this case 0-chains can be ‘trapped’ modulo n at the barycentre of the dual 2-cell. However, by passing to C_\bullet^\dagger this problem is resolved, since then 0-chains at such a vertex may only be assigned coefficient some multiple of the order of that isotropy. \square

The only non-trivial chain group of the relative chain complex of the pair of $C_\bullet(\mathfrak{T})$ and $C_\bullet^\dagger(\mathfrak{T})$ is a torsion group in degree zero. For example, for the SIS \mathfrak{T}^0 associated to a rotationally FLC 2-dimensional tiling T of \mathbb{R}^d , this degree zero group is isomorphic to $\prod_{T_i} \mathbb{Z}/n_i\mathbb{Z}$, where the sum is taken over all rotation classes of tilings T_i of $\Omega(\mathfrak{T}^0)$ with n_i -fold rotational symmetry at the origin (in the highly non-standard case that there exist uncountably many such tilings, one should replace each uncountable product of torsion factors with a countable one). It follows that

$$H_i(\mathfrak{T}) \cong H_i^\dagger(\mathfrak{T}) \cong H^{d-i}(\mathfrak{T}) \cong \check{H}^{d-i}(\Omega(\mathfrak{T}))$$

for $i \neq 0$ and that $H_0(\mathfrak{T})$ is an extension of $\check{H}^2(\Omega(\mathfrak{T}))$ over a torsion group determined by the rotational symmetries of vertices of T . Similar techniques will work in higher dimensions, but in this case the coefficients assigned to higher dimensional cells will need to be restricted. For example, for $d = 3$ one would need to take into account rotational symmetries of patches about 1-cells. One would expect in the general case for there to exist a Zeeman type spectral sequence [39] relating the PE homology to the PE cohomology, and vice versa, in terms of local coefficients determined by the rotational symmetries of the tiling.

Example 3.16. Consider again the SIS \mathfrak{T}_Δ^0 associated to the periodic triangle tiling T of Example 3.13. The chain group

$$C_0(\mathfrak{T}_\Delta^0) \cong \mathbb{Z}v \oplus \mathbb{Z}e \oplus \mathbb{Z}f$$

is generated over \mathbb{Z} by the 3 distinct equivalence classes of 0-cells of \mathcal{T}_Δ , the barycentres v of vertices, e of edges and f faces of T . Since these vertices have rotational symmetry of order 6, 2 and 3, respectively, we have that

$$C_0^\dagger(\mathfrak{T}_\Delta^0) \cong 6\mathbb{Z}v \oplus 2\mathbb{Z}e \oplus 3\mathbb{Z}f.$$

One easily computes the corresponding homology group in degree zero to be $H_0^\dagger(\mathfrak{T}_\Delta^0) \cong \mathbb{Z}$, which is in agreement with the fact that

$$H^2(S^2) \cong \check{H}^2(\Omega(\mathfrak{T}_\Delta^0)) \cong H^2(\mathfrak{T}_\Delta^0) \cong H_0^\dagger(\mathfrak{T}_\Delta^0) \cong \mathbb{Z}.$$

4. PE HOMOLOGY OF HIERARCHICAL TILINGS

In this final section we shall define a method of computation for the PE invariants of a tiling equipped with a certain type of hierarchical structure. It shall be applicable to recognisable substitution tilings of Euclidean space, such as those considered in [1, 4] (see §4.5.1, §4.5.2 and §4.5.3) but also to non-Euclidean ‘combinatorial substitutions’, such as Bowers and Stephenson’s ‘regular pentagonal tilings’ [9] (see §4.5.6). It shall also be applicable to mixed-substitution systems [15] (see §4.5.4). The method is best

illustrated through examples, so we recommend the reader to skip the preliminary details of the general setting on a first reading and to head to §4.5, where the method is applied to a broad range of examples.

4.1. Hierarchical Systems of Tilings. The two main approaches to producing interesting examples of aperiodic tilings, such as the Penrose tilings, are through the cut-and-project method (see [14]) and through tiling substitutions (see [1]); we shall focus on the latter. A substitution rule consists of a finite collection of *prototiles* of \mathbb{R}^d , a rule for subdividing these prototiles and an expanding dilation which, when applied to the subdivided prototiles, defines patches of translates of the original prototiles. By iterating this procedure and inflating, one produces larger and larger patches of tiles, each of which is a translate of one of the originals. A tiling is said to be *admitted by the substitution rule* if every bounded patch is a sub-patch of a translate of some iteratively substituted prototile.

Under certain conditions on the substitution rule, tilings admitted by it exist and, in addition, for each such tiling T_0 there is a *supertiling* T_1 , based on inflated versions of the prototiles, which subdivides to T_0 and is itself a (dilation of an) admitted tiling. So T_0 has a hierarchical structure: there is an infinite list of substitution tilings T_0, T_1, T_2, \dots for which the tiles of T_0 may be grouped to form the supertiles of T_1 , the supertiles of T_1 may be grouped into the super²tiles of T_2 , \dots and so on.

There are several ways in which this process may be generalised. Firstly, one may allow for general rigid motions instead of just translations in the construction, such as in Radin's pinwheel tilings where tiles point in infinitely many directions. Secondly, one may consider hierarchical tilings of more general spaces than of just Euclidean space. An interesting set of examples are given by Bowers and Stephenson's pentagonal tilings [9]. Thirdly, the requirement that the substitute of a prototile has precisely the same support is not entirely necessary, see for example the Penrose kite and dart substitution. Finally, instead of using a single substitution rule to define the supertiles, one may choose a finite list of distinct substitutions and an infinite sequence in which to apply them. In the one dimensional case, these symbolic sequences are often known as S -adic systems. In general dimensions, the tilings that arise are known as mixed or multi-substitution tilings [15]. Passing to the setting of mixed substitutions adds far more generality; for example, every Sturmian word may be defined using a mixed substitution system, whereas the only Sturmian words which may be expressed as purely substitutive tilings are those associated to quadratic irrationals. In contrast to the purely substitutive case, the family of one dimensional mixed substitution tilings exhibit an uncountable number of distinct isomorphism classes of degree one Čech cohomology groups [32]. In a mixed substitution tiling the tiles group together to the supertiles, those into super²tiles, and so on, just as in the purely substitutive case, but now the rules connecting the various levels of the hierarchy are not constant. A general framework which captures this idea is laid out in [28].

To attempt to systematically deal with such a range of examples, we shall take the list of supertilings T_0, T_1, T_2, \dots as a description of the hierarchy of T_0 , instead of some underlying set of substitution rule(s). To begin setting up notation, let \mathcal{T} be a CW poset; the cells of the supertilings will be built out of the cells of \mathcal{T} . Then fix directed

sets Λ (as a directed set of radii) and \mathcal{I} (which shall index the tilings). Suppose that for each $\alpha \in \mathcal{I}$ one has an SIS \mathfrak{T}_α defined over \mathcal{T} and radius poset Λ for which, given $\lambda \in \Lambda$ and $\alpha \leq \beta$ in \mathcal{I} , we have that $\mathfrak{T}_\alpha^\lambda \supseteq \mathfrak{T}_\beta^\lambda$ (so any radius λ internal symmetry of \mathfrak{T}_β is also a radius λ internal symmetry of any \mathfrak{T}_α below it in the hierarchy). Then we may define an SIS \mathfrak{T}_∞ over \mathcal{T} with radius poset $\Lambda \times \mathcal{I}$, where $(\lambda, \alpha) \leq (\mu, \beta)$ in $\Lambda \times \mathcal{I}$ if $\lambda \leq \mu$ and $\alpha \leq \beta$. To do this, we set $\Phi \in (\mathfrak{T}_\infty)^{(\lambda, \alpha)}$ whenever $\Phi \in \mathfrak{T}_\alpha^\lambda$. So a Borel–Moore chain is PE in \mathfrak{T}_∞ if and only if it is PE in some \mathfrak{T}_α .

We wish to specialise this construction of \mathfrak{T}_∞ to the case where each \mathfrak{T}_α corresponds to the internal symmetries of some combinatorial tiling. So suppose that, in addition to the ‘base’ CW poset \mathcal{T} , we have CW posets \mathcal{T}_α for each $\alpha \in \mathcal{I}$, corresponding to the CW posets of the supertilings. We assume that to each cell $a \in \mathcal{T}_\alpha$ there is an associated subcomplex $a_{\mathcal{T}}$ of \mathcal{T} . We demand that $|a_{\mathcal{T}}| \cong B^n$ for an n -cell $a \in \mathcal{T}_\alpha$, and that $a \leq b$ in \mathcal{T}_α if and only if $a_{\mathcal{T}} \subseteq b_{\mathcal{T}}$. In short, each \mathcal{T}_α corresponds to a regular CW decomposition of the space $|\mathcal{T}|$, all of them having \mathcal{T} as a common refinement.

To each of the CW posets \mathcal{T}_α we wish to associate an SIS \mathfrak{T}_α , analogously to the construction of §2.1.2, but over the base CW poset \mathcal{T} . We may want to label cells or to rule out certain morphisms between cells as being non-allowed symmetries, such as, for example, when we wish to consider only translations between patches of a Euclidean tiling, or orientation preserving rigid motions between patches. So we suppose that each \mathcal{T}_α is equipped with a groupoid L_α of allowed morphisms, whose elements are the cells of \mathcal{T}_α and morphisms between them are given by cellular isomorphisms. By a cellular isomorphism between $a, b \in \mathcal{T}_\alpha$, we shall in fact always mean a cellular isomorphism $\Phi: a_{\mathcal{T}} \rightarrow b_{\mathcal{T}}$ in \mathcal{T} which induces a cellular isomorphism $\Phi': \bar{a} \rightarrow \bar{b}$ in \mathcal{T}_α (and shall usually drop the relevant notation indicating this). When we later introduce our homology calculations, we shall always assume that our coefficient group G has division by the order of the isotropy groups of each L_α .

Given a cell $a \in \mathcal{T}$ and $n \in \mathbb{N}_0$, we let $P_\alpha(a, n)$ be the closure of the set of cells in \mathcal{T}_α which contain a cell of \mathcal{T} in their interiors within combinatorial distance n of a (see §2.1.2 for a notion of combinatorial distance between cells). So we are measuring sizes of patches using the cells of \mathcal{T} , not of \mathcal{T}_α . Then given $a, b \in \mathcal{T}$, let $\Phi \in (\mathfrak{T}_\alpha)_{a,b}^n$ if and only if Φ is the restriction to $\overline{\text{St}(a)}$ of some cellular isomorphism $\tilde{\Phi}$ between $P_\alpha(a, n)$ and $P_\alpha(b, n)$ which is permitted by the allowed morphisms of L_α .

We shall call the collection of the above data a *hierarchical system* if for $\alpha \leq \beta$ we have that $\mathfrak{T}_\alpha \supseteq \mathfrak{T}_\beta$, and we collect this data into the SIS \mathfrak{T}_∞ as constructed in greater generality above. This condition may be thought of as saying that, for $\alpha \leq \beta$, the radius n patch of tiles of \mathcal{T}_α at a cell $a \in \mathcal{T}$ is determined by the n -patch of a in \mathcal{T}_β ; compare to the case of an ordered list T_0, T_1, T_2, \dots of supertilings arising from a substitution as discussed above.

In many cases of interest, there is a local rule for ‘undoing’ the subdivision, so that a patch $P_\beta(c, n)$ may always be determined up to equivalence using only the information of the patch $P_\alpha(c, m)$ below it in the hierarchy, for possibly larger radius m . This case would correspond to a *recognisable* substitution, and in the language here this simply means that each \mathfrak{T}_α is tail-equivalent to \mathfrak{T}_∞ (and in the language of tiling theory that the tilings corresponding to \mathfrak{T}_α are MLD [34, Chapter 1, §3]). So in this case a chain

is PE in *some* \mathfrak{T}_α if and only if it is PE in *any given* \mathfrak{T}_α , and calculation of invariants of \mathcal{T}_∞ are tantamount to calculating the invariants of any \mathfrak{T}_α in the hierarchy. In the non-recognisable case, moving up the hierarchy adds more information, so that a tiling \mathcal{T}_β may ‘know’ something that a tiling \mathcal{T}_α does not, see the solenoid example §4.5.5.

4.2. Growth of Cells. To apply the main theorem of this section, which determines a method of computation of the PE homology of \mathfrak{T}_∞ , we shall need the cells of each \mathcal{T}_α to become large relative to the cells of \mathcal{T} as α increases. The precise condition that we require is defined here; we note that this condition will hold for any of the examples that we are actually interested in here, such as polytopal (mixed) substitution tilings of Euclidean space.

For each $\alpha \in \mathcal{I}$, $k \in \{0, \dots, d\}$ and $x \in \mathcal{T}$ contained in the k -skeleton of \mathfrak{T}_α , let $N^{\alpha,k}(x) \subseteq \mathcal{T}_\alpha$ be a subset of cells. We think of the cells of $N^{\alpha,k}(x)$ as being ‘near to x ’, so we shall always assume that if $x \in a_\mathcal{T}$ for $a \in \mathcal{T}_\alpha$ then $a \in N^{\alpha,k}(x)$. Fix a coefficient ring G which has division by the order of isotropy for each L_α . For $\sigma \in C_n^{\text{BM}}(\mathcal{T}; G)$, we write $\sigma \in N_n^{\alpha,k}$ to mean that σ is supported on the subcomplex corresponding to the k -skeleton of \mathfrak{T}_α and is such that, for any n -cell $x \in \mathcal{T}$ of this subcomplex and any combinatorial isomorphism $\Phi: X \rightarrow Y$ between subcomplexes of \mathfrak{T}_α (permitted by the allowed morphisms of L_α) with $N^{\alpha,k}(x) \subseteq X$, we have that $\sigma(\omega_x) = \sigma(\Phi_*(\omega_x))$ for an orientation ω_x of x . We think of the chains of $N_n^{\alpha,k}$ as the chains which assign coefficients to cells of \mathcal{T} in a way which only depends locally on the cells of \mathfrak{T}_α .

Definition 4.1. We shall say that the hierarchical system \mathfrak{T}_∞ is *expansive* if, for any $\alpha \in \mathcal{I}$ and $m \in \mathbb{N}_0$, we can choose $\beta \geq \alpha$ and $N^{\beta,k}(x)$, as above, so that:

- (1) Each radius m patch $P_\beta(x, m) \subseteq N^{\beta,d}(x)$.
- (2) For each k -cell $a \in \mathcal{T}_\beta$, there exists some k -cell $x \in a_\mathcal{T}$ for which $N^{\beta,k}(x) \subseteq \overline{\text{St}(a)}$.
- (3) For any $\sigma \in N_n^{\beta,k}$ with $n < k$ and $\partial(\sigma)$ supported on the $(k-1)$ -skeleton of \mathcal{T}_β , there exists some PE chain $\tau \in C_{n+1}(\mathfrak{T}_\infty)$ for which $\sigma + \partial(\tau) \in N_n^{\beta,k-1}$.

Example 4.2. Let \mathcal{K} be a CW decomposition of \mathbb{R}^d . We shall call \mathcal{K} *polytopal* if each cell $a \in \mathcal{K}$ may be assigned a barycentre v_a for which:

- (1) Each closed cell a of \mathcal{K} is realised as the geometric simplicial complex of simplices the convex hulls of $\{v_{a_1}, v_{a_2}, \dots, v_a\}$ for $\{a_1, a_2, \dots, a\} \in (\mathcal{F}(\mathcal{K}))_\Delta$.
- (2) Whenever Φ is a cellular rigid motion between closed cells a, b of \mathcal{K} , then Φ maps v_a to v_b .

For example, if the cells of \mathcal{K} are convex then \mathcal{K} is polytopal, one may choose the barycentre v_c to be the centre of mass of c in its supporting hyperplane.

Suppose that our hierarchical system \mathfrak{T}_∞ is based on CW posets \mathcal{T} and \mathcal{T}_α of polytopal CW decompositions of \mathbb{R}^d , that the allowed cellular isomorphisms L_α are based upon rigid motions and that:

- (1) There exists some r for which each cell $a \in \mathcal{T}$ is fully contained in some r -ball.
- (2) For each $R > 0$, for sufficiently large $\alpha \in \mathcal{I}$ the barycentres v_a of cells $a \in \mathcal{T}_\alpha$ satisfy $|v_a - v_b| \geq R$ for $a \neq b$.

Then we say that \mathfrak{T}_∞ is based on polytopal and growing cells. In this case, the hierarchical system is expansive. The polytopal condition allows one to use the barycentric subdivision to define regions surrounding the cells which deformation retract to lower dimensional skeleta. The sets $N^{\alpha,k}(x)$ may be defined using these regions, and any potential fringe effects associated to making the process cellular may be avoided by choosing slightly larger regions about cells as k decreases. In the case of non-trivial isotropy, one may need to invoke the divisibility of the coefficient group in the construction of the boundary chains τ , analogously to in the proof of Proposition 3.6.

4.3. The Method of Computation. Fix a coefficient ring G which, as previously stated, we will assume to have division by the order of isotropy of allowed cellular isomorphisms of each L_α . For $\alpha \in \mathcal{I}$ we define the *approximant chain complex* A_\bullet^α as follows. Let $A_n^\alpha \leq C_n(\mathcal{T}_\alpha)$ consist of the Borel–Moore n –chains of \mathcal{T}_α which assign coefficients to cells in a way which depends only locally on stars of cells of \mathfrak{T}_α . That is, $\sigma \in A_n^\alpha$ if and only if for any cellular isomorphism $\Phi: \text{St}(a) \rightarrow \text{St}(b)$ permitted by L_α sending a^n to b^n , we have that $\sigma(\omega_a) = \sigma(\Phi_*(\omega_a))$ for an orientation ω_a of a . The cellular boundary map makes each A_\bullet^α a chain complex. We call the corresponding homology $H_\bullet A_\alpha := H(A_\bullet^\alpha)$ an *approximant homology*.

We now wish to define, for each $\alpha \leq \beta$ in \mathcal{I} , a homomorphism $s_\alpha^\beta: H_\bullet A_\alpha \rightarrow H_\bullet A_\beta$ so that the approximant homologies fit into a directed system over \mathcal{I} . So let σ be a cycle of A_n^α . We may identify σ with a cycle of \mathcal{T} , supported on the subcomplex corresponding to the n –skeleton of \mathcal{T}_α . Since this chain assigns values to cells based only immediate local data in \mathfrak{T}_α , it still only depends on local data in \mathfrak{T}_β by our assumption of the tilings forming a hierarchical system. Unfortunately, σ may not be supported on the n –skeleton of \mathcal{T}_β .

The idea is to now push σ back to the n –skeleton using consistent choices for each equivalent cell. Suppose the maximal dimension of cell is d . If $n = d$, then σ is already supported on the n –skeleton. Otherwise, for each equivalence class of d –cell of \mathcal{T}_β , choose a representative a . Restrict σ to $a_{\mathcal{T}}$ and choose an $(n+1)$ –chain τ_a supported on the interior of a for which $\sigma|_{a_{\mathcal{T}}} + \partial(\tau_a)$ is supported on the boundary of a . One may now copy the chain τ_a to each equivalent cell using the cellular isomorphisms permitted by L_β . In the case of non-trivial cell isotropy, one needs to invoke the divisibility of the coefficient group to ‘average’ the boundary chain over the cells’ symmetries, as in the proof of Proposition 3.6. By repeating this operation for each equivalence class of cell one defines a chain τ , which only depends locally on stars of cells in \mathfrak{T}_β , for which $\sigma + \partial(\tau)$ is supported on the $(d-1)$ –skeleton. Continue this operation iteratively down the skeleta until the original chain is seen to be homologous to one supported on the n –skeleton of \mathcal{T}_β . We denote the corresponding cycle of A_n^β by $s_\alpha^\beta(\sigma)$. It shall follow from the proof of Theorem 4.3 that this procedure induces well-defined homomorphisms, called *connecting maps*, between the approximant homology groups, making them a directed system over \mathcal{I} .

To summarise, the construction goes as follows:

- (1) For each $\alpha \in \mathcal{I}$, enumerate the list of star-patches of cells of \mathcal{T}_α , up to equivalence.

- (2) Define the approximant chain groups A_n^α to be the groups freely generated by the equivalence classes of star-patches of n -cells of \mathcal{T}_α (which do not possess self-symmetries reversing orientations).
- (3) Compute the corresponding approximant homology $H_\bullet A_\alpha$.
- (4) For $\alpha \leq \beta$, connecting maps $s_\alpha^\beta: H_\bullet A_\alpha \rightarrow H_\bullet A_\beta$ are defined as follows. Given a cycle $\sigma \in A_n^\alpha$, firstly canonically identify it with the cycle of the finer complex \mathcal{T} . Iteratively push σ down the skeleta of \mathcal{T}_β by making a choice of $(n+1)$ -chain for each equivalence class of star-patch of \mathcal{T}_β (which is invariant under any potential non-trivial cell isotropy). The resulting cycle of $H_\bullet A_\beta$ is defined to be $s_\alpha^\beta(\sigma)$.
- (5) Compute the direct limit of the directed system of approximant homologies $H_\bullet A_\alpha$ and substitution homomorphisms s_α^β between them.

Notice that for a Borel–Moore chain of \mathcal{T} which is PE to radius n in \mathfrak{T}_α , the same is true of its boundary. Denote by B_\bullet^α the chain complex of chains which are PE to radius 0 in \mathfrak{T}_α , that is, the complex of chains which depend only on the cells of their immediate neighbourhood in \mathfrak{T}_α . By the assumption of the \mathfrak{T}_α forming a hierarchical system, we have an inclusion of chain complexes

$$\iota_\alpha^\beta: B_\bullet^\alpha \hookrightarrow B_\bullet^\beta$$

for $\alpha \leq \beta$. The corresponding union of chain complexes shall be denoted

$$B_\bullet := \bigcup_{\alpha \in \mathcal{I}} B_\bullet^\alpha \cong \varinjlim_{\mathcal{I}} (B_\bullet^\alpha, \iota_\alpha^\beta),$$

which is the subcomplex of $C_\bullet(\mathfrak{T}_\infty)$ of chains which are PE to radius 0 in some \mathfrak{T}_α .

Theorem 4.3. There exist canonical isomorphisms $H_\bullet A_\alpha \cong H(B_\bullet^\alpha)$ for $\alpha \in \mathcal{I}$ which together define an isomorphism between the diagrams $(H_\bullet A_\alpha, s_\alpha^\beta)$ and $(H(B_\bullet^\alpha), (\iota_\alpha^\beta)_*)$. If the hierarchy is expansive, then the canonical inclusion of chain complexes

$$i: B_\bullet \rightarrow C_\bullet(\mathfrak{T}_\infty)$$

is a quasi-isomorphism. In particular, in this case (such as when \mathfrak{T}_∞ is Euclidean and based upon on polytopal and growing cells), the method outlined above computes the PE homology of \mathfrak{T}_∞ :

$$H_\bullet(\mathfrak{T}_\infty) \cong \varinjlim_{\mathcal{I}} (H_\bullet A_\alpha, s_\alpha^\beta).$$

Proof. The isomorphisms $H_n A_\alpha \cong H_n(B_\bullet^\alpha)$ are induced from the canonical chain isomorphisms $C_\bullet(\mathcal{T}_\alpha) \cong H_\bullet(\mathcal{T}_\alpha^\bullet, \mathcal{T}_\alpha^{\bullet-1})$. To ensure that this induces a quasi-isomorphism upon restricting to chains which are PE to radius 0 in each \mathfrak{T}_α , one needs to implement the assumption of the coefficient group having division by the order of the isotropy of cells, analogously to the proof of Proposition 3.6.

So now suppose that the hierarchical system is expansive. We shall firstly show that the inclusion of chain complexes i induces a surjective map on homology. Let σ be a cycle of $C_n(\mathfrak{T}_\infty)$, so σ is PE to some radius m in some \mathfrak{T}_α . By assumption, there exists $\beta \geq \alpha$ and choices $N^{\beta,k}(x)$ of neighbours to cells x for which conditions 1–3 of Definition 4.1 are satisfied. Condition 1 implies that $\sigma \in N_n^{\beta,d}$. With repeated application of condition 3, we may find a PE $(n+1)$ -chain τ for which $\sigma + \partial(\tau) \in N_n^{\beta,n}$. By condition

2, for each n -cell $a \in \mathcal{T}_\alpha$ there exists an n -cell $x \in a_{\mathcal{T}}$ for which $N^{\beta,n}(x) \subseteq \overline{\text{St}(a)}$. Since the value of $\sigma + \partial(\tau)$ on x determines the value of $\sigma + \partial(\tau)$ on $a_{\mathcal{T}}$, and $\sigma + \partial(\tau) \in N_n^{\beta,n}$, we see that $\sigma + \partial(\tau) \in B_\bullet^\beta$. Hence, the inclusion homomorphism i induces a surjective map on homology.

To show surjectivity, suppose that $\sigma \in B_n$ and that $\sigma = \partial(\tau)$ for $\tau \in C_{n+1}(\mathfrak{T}_\infty)$. There exists some $\alpha \in \mathcal{I}$ and $m \in \mathbb{N}_0$ for which σ is PE to radius 0 in \mathfrak{T}_α and τ is PE to radius m in \mathfrak{T}_α . By expansivity, we may pick $\beta \geq \alpha$ for which conditions 1–3 of Definition 4.1 are satisfied. Note that there exists some chain $\tau' \in B_{n+1}^\beta$ for which $\sigma' := \sigma + \partial(\tau')$ is supported on \mathcal{T}_β^m . Since $\sigma' \simeq \sigma$ in $H_n(B_\bullet^\beta)$ and $\tau + \tau'$ is still PE to radius m in \mathfrak{T}_β , it is sufficient to show that $\tau + \tau'$ is homologous to an element of B_{n+1} . Similarly to the above, we have that $\tau + \tau' \in N_{n+1}^{\beta,d}$ and, since $\partial(\tau + \tau') = \sigma'$ is supported on the subcomplex of \mathcal{T} corresponding to the n -skeleton of \mathcal{T}_β , we may make repeated use of condition 3 to find an element $\tau'' \in N_{n+1}^{\beta,n+1}$ which is homologous to $\tau + \tau'$. Again, by condition 2 of Definition 4.1, it follows that $\tau'' - \tau' \in B_{n+1}^\beta$. Since $\partial(\tau'' - \tau') = \partial(\tau) = \sigma$, the inclusion of chain complexes i is injective on homology. \square

4.4. Discussion of Method, and its Relation to Others. When one has a handle on the local combinatorics for each \mathfrak{T}_α , and the rules connecting them between neighbouring levels of the hierarchy, the above theorem allows one to compute the PE homology of \mathfrak{T}_∞ . For example, in the case that \mathfrak{T}_∞ corresponds to the hierarchy of a Euclidean substitution tiling, the combinatorics at each level and the rules for passing between them are constant, and so there is essentially only one approximant homology group and one connecting map to be determined. More generally (see Example 4.2), the method will apply to any (recognisable) Euclidean hierarchical tiling based on polytopal and growing cells, such as for a polytopal mixed substitution tiling.

The method of computation seems to be quite efficient. Indeed, without determining that the tiling ‘forces the border’ (see [1, §4]), the list of star-patches is the minimal amount of collaring information required to determine the immediate neighbourhood of a point of the tiling. This list of star-patches, and their incidences, define the approximant chain complexes in a conveniently direct way.

The approach seems to be closely related to that of Barge, Diamond, Hunton and Sadun [4], although making the connection precise seems to be somewhat technical. The argument that the two methods produce the same direct limits would go as follows. Take the ‘dual stratification’ $S_0 \subset S_1 \subset \dots \subset K_t$ of the BDHS approximant (where t is small with respect to the size of the tiles) where each S_i corresponds to the union of k -cell flaps with $k \geq d - i$ (see [4, §4]). Then, using a Poincaré duality type argument, so long as the ‘cell-flaps’ and their incidences match the combinatorics of the tiling, one should be able to show that the (regraded) approximant chain complex $A_{d-\bullet}^0$ is isomorphic to the relative complex $H^\bullet(S_\bullet, S_{\bullet-1})$, and that this has cohomology isomorphic to $H^{d-\bullet}(K_t)$, by the vanishing of the relative groups $H^i(S_j, S_{j-1})$ for $i \neq j$.

These computations also seem to be related to a method developed in [16], where Gonçalves used the duals of the approximant complexes A_\bullet^α defined here to determine approximant groups to the K -theory associated to the stable equivalence relation of a substitution tiling. It was observed there that there appears to be a duality between

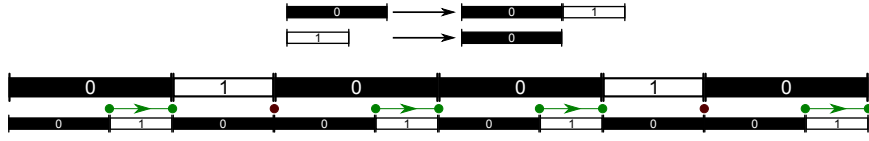


FIGURE 4.1. Fibonacci Tiling

the resulting direct limit groups and the K -theory of the tiling space. For a concrete relation one would also need to compare the connecting maps between approximant complexes defined in [16] to those constructed here.

Finally, we wish to discuss the computations in the presence of rotational symmetries. The method outlined above will compute the PE homology $H_\bullet(\mathfrak{T}_\infty)$, but over non-divisible coefficients these groups need not be Poincaré dual to the PE cohomology (see Example 3.13 and §4.5.3). However, one may modify the method, by restricting the value of chains at cells whose star-patches possess non-trivial symmetries. This allows one to compute the modified PE homology $H_\bullet^\dagger(\mathfrak{T}^0)$ (see §3.4), which is Poincaré dual to the PE cohomology, see the computation of §4.5.3.

4.5. Example Computations.

4.5.1. Fibonacci Tilings. Fibonacci tilings are tilings of \mathbb{R}^1 of two prototiles, which we shall call here 0 and 1, which are intervals of lengths the golden ratio φ and 1, respectively. They are examples of Sturmian words, but may also be constructed via the recognisable substitution $0 \mapsto 01$, $1 \mapsto 0$. A tiling T admitted by this substitution defines a cellular decomposition \mathcal{T} of the real line and an infinite sequence of tilings $T = T_0, T_1, \dots$ for which the substitution rule subdivides T_{i+1} to T_i . The tiles of T_i are based on inflations of the original tiles of T_0 by a scaling factor of φ^i .

We shall compute the PE homology groups associated to the translational hull of Fibonacci tilings using the method outlined above. To determine the approximant chain complexes A_n^i , we firstly enumerate the equivalence classes of star-patches up to translation. They are given by the ‘vertex types’ 0.1, 1.0 and 0.0, and the ‘edge types’ are given by the equivalence classes associated to the two distinct prototiles 0 and 1. So each approximant chain complex is of the form

$$0 \leftarrow \mathbb{Z}^3 \xleftarrow{\partial_1} \mathbb{Z}^2 \leftarrow 0.$$

Orient the 1-cells to point to the right and define the 0-chains $\mathbb{1}(0.1)_i$, $\mathbb{1}(1.0)_i$, $\mathbb{1}(0.0)_i$ and 1-chains $\mathbb{1}(0)_i$ and $\mathbb{1}(1)_i$ to be the canonical indicator chains associated to each of the edge and vertex types in the level i tiling. Then the degree one boundary maps are given by:

$$\begin{aligned} \partial_1(\mathbb{1}(0)_i) &= \mathbb{1}(0.1)_i - \mathbb{1}(1.0)_i \\ \partial_1(\mathbb{1}(1)_i) &= \mathbb{1}(1.0)_i - \mathbb{1}(0.1)_i \end{aligned}$$

So $H_0A_i \cong \mathbb{Z}^2$ and is freely generated by the indicator chains $a_i := \mathbb{1}(0.1)_i$ and $b_i := \mathbb{1}(0.0)_i$. The degree one approximant homology group $H_1A_i \cong \mathbb{Z}$ is generated by the fundamental class $\Gamma_i := \mathbb{1}(0)_i + \mathbb{1}(1)_i$.

We now calculate the connecting maps s_i^{i+1} between these approximant groups. Recall that to define these homomorphisms, given a cycle $\sigma \in H_nA_i$ we consider it as a chain depending only on its immediate neighbourhood in the tiling T_{i+1} next up the hierarchy. One then pushes the chain back to the n -skeleton to define $s_i^{i+1}(\sigma)$. Clearly the fundamental class Γ_i is mapped to Γ_{i+1} . To calculate the connecting map in degree zero, firstly consider the generator a_i of H_0A_i . The 0.1 vertices of T_i are interior points of the 0 tiles of T_{i+1} , which subdivides to 01 in T_i . We may push this chain to the 0-skeleton of T_{i+1} in a local way by, for example, moving it to the right-hand endpoint of each 0 tile of T_{i+1} (see Figure 4.1), so we see that $s_i^{i+1}(a_i) \simeq \mathbb{1}(0.1)_{i+1} + \mathbb{1}(0.0)_{i+1} = a_{i+1} + b_{i+1}$. For the generator b_i , note that each 0.0 vertex of T_i is found precisely on a 1.0 vertex of T_{i+1} . So we see that $s_i^{i+1}(b_i) = \mathbb{1}(1.0)_{i+1} \simeq \mathbb{1}(0.1)_{i+1} = a_{i+1}$. Then the connecting maps $s_i^{i+1}: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ send the generators $a_i \mapsto a_{i+1} + b_{i+1}$ and $b_i \mapsto a_{i+1}$, which are isomorphisms. So we have confirmed that:

$$\begin{aligned} \check{H}^0(\Omega^1) &\cong H^0(\mathfrak{T}^1) \cong H_1(\mathfrak{T}^1) \cong \varinjlim(\mathbb{Z}, s_i^{i+1}) \cong \mathbb{Z} \\ \check{H}^1(\Omega^1) &\cong H^1(\mathfrak{T}^1) \cong H_0(\mathfrak{T}^1) \cong \varinjlim(\mathbb{Z}^2, s_i^{i+1}) \cong \mathbb{Z}^2 \end{aligned}$$

The first isomorphisms come from the fact that the Čech cohomology of the translational hull Ω^1 is isomorphic to the PE cohomology (see [33] or §3.2), the second from the PE Poincaré duality of Theorem 3.11 and the penultimate isomorphism from Theorem 4.3 and the recognisability of the substitution.

4.5.2. Thue–Morse Tilings. The Thue–Morse tilings are produced via the recognisable substitution $0 \mapsto 01$ and $1 \mapsto 10$. In this case, the vertex types are given by 0.0, 0.1, 1.0 and 1.1, and the edge types by 0 and 1. The boundary maps have the same description as the above example, so the approximant homology groups are $H_0A_i \cong \mathbb{Z}^3$ and $H_1A_i \cong \mathbb{Z}$. Of course, H_1A_i is generated by the fundamental class $\Gamma_i = \mathbb{1}(0)_i + \mathbb{1}(1)_i$ and $s_i^{i+1}(\Gamma_i) = \Gamma_{i+1}$. For the degree zero calculation, firstly note that H_0A_i is generated by the 0-chains $a_i := \mathbb{1}(0.1)_i$, $b_i := \mathbb{1}(0.0)_i$ and $c_i := \mathbb{1}(1.1)_i$. The 0.1 vertices of T_i are found precisely at the centres of 0 tiles and the 1.1 vertices of T_{i+1} . Shifting the chain from the centres of the 0 tiles to the right and back onto the 0-skeleton, we see that $s_i^{i+1}(a_i) = (\mathbb{1}(0.1)_{i+1} + \mathbb{1}(0.0)_{i+1}) + \mathbb{1}(1.1)_{i+1} = a_{i+1} + b_{i+1} + c_{i+1}$. The 0.0 vertices of T_i lie precisely on the 1.0 vertices of T_{i+1} so $s_i^{i+1}(b_i) = \mathbb{1}(1.0)_{i+1} \simeq a_{i+1}$. Similarly, the 1.1 vertices of T_i lie precisely on the 0.1 vertices of T_{i+1} , so $s_i^{i+1}(c_i) = \mathbb{1}(0.1)_{i+1} = a_{i+1}$. In summation, the connecting maps $s_i^{i+1}: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ are given by:

$$\begin{aligned} a_i &\mapsto a_{i+1} + b_{i+1} + c_{i+1} \\ b_i &\mapsto a_{i+1} \\ c_i &\mapsto a_{i+1} \end{aligned}$$

This linear map has eigenvectors with eigenvalues 0, -1 and 2 , although they only span an index 3 sublattice of \mathbb{Z}^3 . With some further calculation one may evaluate the direct

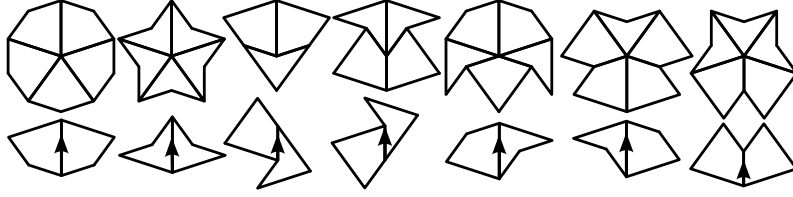


FIGURE 4.2. Vertex and Edge Types of the Penrose Kite and Dart Tilings

limit as being isomorphic to $\mathbb{Z} \oplus \mathbb{Z}[1/2]$, so:

$$\begin{aligned}\check{H}^0(\Omega^1) &\cong H_1(\mathfrak{T}^1) \cong \varinjlim(\mathbb{Z}, s_i^{i+1}) \cong \mathbb{Z} \\ \check{H}^1(\Omega^1) &\cong H_0(\mathfrak{T}^1) \cong \varinjlim(\mathbb{Z}^3, s_i^{i+1}) \cong \mathbb{Z} \oplus \mathbb{Z}[1/2]\end{aligned}$$

4.5.3. *Penrose Kite and Dart Tilings.* We consider now Penrose's famous kite and dart tilings of the plane. Just as for the Fibonacci tilings, this example may be produced via a cut-and-project scheme or via a substitution rule. Since the substitution rule for the kite and dart tiles does not perfectly decompose the support of a supertile into its constituent tiles, we take the underlying complex \mathcal{T} to be given by a tiling of Robinson triangles.

For this example, we shall allow patches to be compared using orientation preserving rigid motions, so that a Borel–Moore chain $\sigma \in H_n^{\text{BM}}(\mathcal{T})$ is PE in this setup if and only if, for sufficiently large radius r , the value of σ at an oriented n -cell c depends only on the patch of tiles within radius r of c , up to equivalence of rigid motion. The homology calculations in the case where we compare patches only up to translation are determined by Poincaré duality, which we provide here for reference (see [1]):

$$\begin{aligned}\check{H}^0(\Omega^1) &\cong H^0(\mathfrak{T}^1) \cong H_2(\mathfrak{T}^1) \cong \mathbb{Z} \\ \check{H}^1(\Omega^1) &\cong H^1(\mathfrak{T}^1) \cong H_1(\mathfrak{T}^1) \cong \mathbb{Z}^5 \\ \check{H}^2(\Omega^1) &\cong H^2(\mathfrak{T}^1) \cong H_0(\mathfrak{T}^1) \cong \mathbb{Z}^8\end{aligned}$$

To begin calculation, we must firstly enumerate the list of star-patches of cells in the tiling up to rigid motion. It turns out that there are 7 such star-patches at vertices (named sun, star, ace, deuce, jack, queen and king in Conway's notation) and there are 7 ways for tiles to meet along an edge, which we shall denote by $E1$ – $E7$; see Figure 4.2 where the vertex and edge types are listed in these respective orders. Of course, the two face types are given by the two types of tiles, kites and darts. So the approximant chain complexes are of the form

$$0 \leftarrow \mathbb{Z}^7 \xleftarrow{\partial_1} \mathbb{Z}^7 \xleftarrow{\partial_2} \mathbb{Z}^2 \leftarrow 0.$$

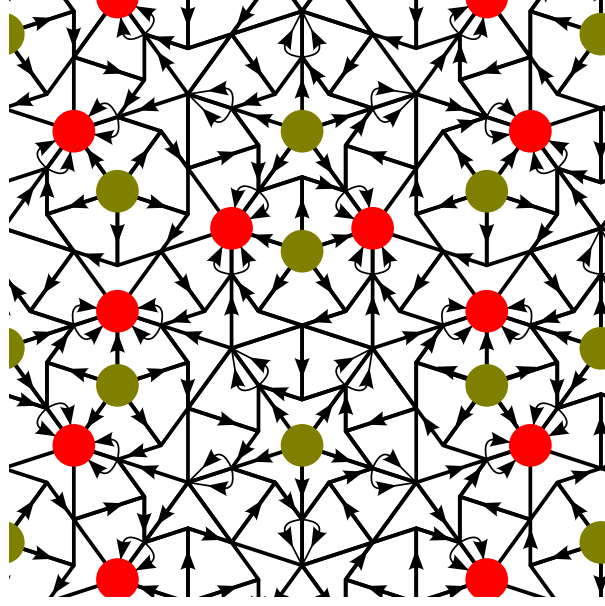


FIGURE 4.3. Torsion Element t_0 with $5t_0 + \partial_1(-\mathbb{1}(E1)_0 + \mathbb{1}(E2)_0 - \mathbb{1}(E4)_0 - 2 \cdot \mathbb{1}(E7)_0) = 0$.

The ∂_1 boundary map, with respect to the bases of vertex and edge types ordered as in Figure 4.2, is represented as a matrix by:

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & -2 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

The first column, for example, is $(5, 0, -1, 0, 1, -1, 0)^T$ since at any sun vertex there are 5 incoming $E1$ edges, at an ace there is an outgoing $E1$, at a jack there is an incoming $E1$ and at a queen there is an outgoing $E1$.

Some simple calculations using the Smith normal form show that the approximant groups $H_0A_i \cong \mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}$, with basis of the free part generated by a_i as the indicator of sun vertex types and b_i the indicator of star vertex types. The 5-torsion is generated by the element $t_i = a_i + b_i - c_i$, where c_i is the indicator of queen vertex types. The torsion element t_0 is illustrated in Figure 4.3, where it is shown pictorially that $5t_0$ is nullhomologous via the boundary of $-\mathbb{1}(E1)_0 + \mathbb{1}(E2)_0 - \mathbb{1}(E4)_0 - 2 \cdot \mathbb{1}(E7)_0$.

To calculate the degree zero connecting maps, note that the sun vertices of T_i lie precisely at the star, queen and king vertices of T_{i+1} , the star vertices at the sun vertices of T_{i+1} and the queen vertices at the deuce vertices of T_{i+1} . Some simple calculations

then show that:

$$\begin{aligned} a_i &\mapsto 3a_{i+1} - b_{i+1} + 2t_{i+1} \\ b_i &\mapsto a_{i+1} \\ t_i &\mapsto t_{i+1} \end{aligned}$$

Since each connecting map is an isomorphism, we conclude that

$$H_0(\mathfrak{T}^0) \cong \varinjlim (\mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}, s_i^{i+1}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}.$$

To calculate using the modified complexes, so as to restore Poincaré duality, one uses the approximant chain complexes

$$0 \leftarrow 5\mathbb{Z} \oplus 5\mathbb{Z} \oplus \mathbb{Z}^5 \xleftarrow{\partial_1} \mathbb{Z}^7 \xleftarrow{\partial_2} \mathbb{Z}^2 \leftarrow 0,$$

where the degree zero group is the subgroup of A_0^i which restricts the coefficients on the sun and star vertices to multiples of 5, since these vertices have 5-fold rotational symmetry and the other vertices have trivial rotational symmetry. We calculate the modified approximant homology groups in degree zero as \mathbb{Z}^2 and the connecting maps as isomorphisms, which confirms that

$$\check{H}^2(\Omega^0) \cong H^2(\mathfrak{T}^0) \cong H_0^\dagger(\mathfrak{T}^0) \cong \varinjlim (\mathbb{Z}^2, s_i^{i+1}) \cong \mathbb{Z}^2.$$

The inclusion of chain complexes $C_\bullet^\dagger(\mathfrak{T}^0) \hookrightarrow C_\bullet(\mathfrak{T}^0)$ induces the following short exact sequence

$$\begin{aligned} (0 \rightarrow [H_0^\dagger(\mathfrak{T}^0) \cong \check{H}^2(\Omega_T^0)] \rightarrow H_0(\mathfrak{T}^0) \rightarrow (\mathbb{Z}/5\mathbb{Z})^2 \rightarrow 0) \cong \\ 0 \rightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z} \xrightarrow{g} (\mathbb{Z}/5\mathbb{Z})^2 \rightarrow 0, \end{aligned}$$

where $f(a, b) = (a + 2b, a - 3b, [4a + 3b]_5)$ and $g(a, b, [c]_5) = ([a + c]_5, [b + c]_5)$, exhibiting the failure of PE Poincaré duality in this example.

For degree one the situation is illustrated in Figure 4.4. One may calculate that each degree one approximant group $H_1 A_i \cong \mathbb{Z}$ is generated by the 1-chain $e_i := \mathbb{1}(E3)_i + \mathbb{1}(E4)_i$. That is, the degree one approximant groups are freely generated by the 1-cycles which trail the bottoms of the level i dart supertiles. Such a cycle is illustrated in red in Figure 4.4, along with the analogous green cycle next up the hierarchy (although we have removed any indication of orientations to decrease clutter). As shown in the figure, the two chains are related by the boundary of the indicator 2-chain associated to the dart tiles of the lower hierarchy. So we see that $s_i^{i+1}(e_i) = -e_{i+1}$ (note the reversal of orientations) and, in particular, the connecting maps in degree one are isomorphisms, which confirms that

$$\check{H}^1(\Omega^0) \cong H^1(\mathfrak{T}^0) \cong H_1(\mathfrak{T}^0) \cong \mathbb{Z}.$$

4.5.4. Arnoux–Rauzy Words. Since the previous examples were based upon purely substitutive systems, the approximant homology groups and connecting maps between them were canonically isomorphic between adjacent levels of the hierarchy. We consider now a *mixed* substitution system, where one has a set of substitutions and an infinite sequence in which to apply them.

Our examples here will be the Arnoux–Rauzy words, originally introduced in [3] as a generalisation of Sturmian words. Let $k \in \mathbb{N}_{\geq 2}$. The Arnoux–Rauzy substitutions

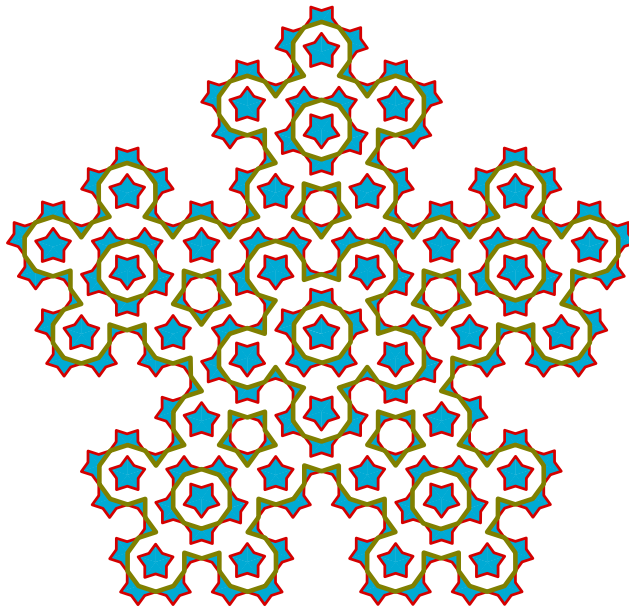


FIGURE 4.4.

are defined over the alphabet $\mathcal{A}_k = \{1, 2, \dots, k\}$ and the k different substitutions ρ_i are given by $\rho_i(j) = ji$ for $i \neq j$ and $\rho_i(i) = i$. Fix an infinite sequence $(n_i)_i = (n_0, n_1, \dots) \in \mathcal{A}_k^{\mathbb{N}_0}$ for which each element of \mathcal{A}_k occurs infinitely often. Then there exist bi-infinite *Arnoux–Rauzy words* for which every finite sub-word is contained in some translate of a ‘supertile’ $\rho_{n_0} \circ \rho_{n_1} \circ \dots \circ \rho_{n_l}(i)$. We may consider such a word as defining a tiling of labelled unit intervals of \mathbb{R}^1 . Everything is recognisable, so for such a tiling T_0 , one may uniquely group the tiles to a tiling T_1 of tiles of labelled intervals (although of different lengths) so that the substitution ρ_{n_0} decomposes T_1 to T_0 . The process may be repeated, so one in fact obtains an infinite hierarchy of tilings, the supertiles of which become arbitrarily large as one passes up the hierarchy.

The two letter words of T_i are the elements of $\mathcal{A}_k \times \mathcal{A}_k$ with at least one occurrence of n_i . So it is easy to see that $H_0 A_i \cong \mathbb{Z}^k$ is freely generated by the indicator 0-chains of vertices of the form $n_i \cdot j$ where $j \in \mathcal{A}_k$ is arbitrary. A simple calculation shows that, with this choice of basis, the connecting map s_i^{i+1} is the unimodular matrix M_i given by the identity matrix with a column of 1’s down the n_i^{th} column which, incidentally, is the incidence matrix of the substitution.

It follows that the degree one Čech cohomology of the tiling space of the Arnoux–Rauzy words associated to any given sequence $(n_i)_i \in \mathcal{A}_k^{\mathbb{N}_0}$ is

$$\check{H}^1(\Omega^1) \cong H^1(\mathfrak{T}_\infty) \cong H_0(\mathfrak{T}_\infty) \cong \varinjlim (\mathbb{Z}^k \xrightarrow{M_0} \mathbb{Z}^k \xrightarrow{M_1} \mathbb{Z}^k \xrightarrow{M_2} \dots) \cong \mathbb{Z}^k.$$

It is interesting to note that the matrices above are related to continued fraction algorithms. For example, for the $k = 2$ case, the Arnoux–Rauzy words are precisely the Sturmian words. To an irrational α , the sequence $(n_i)_i$ is chosen according to the continued fraction algorithm for α (see [13, §3.2]) and the sequence of matrices M_i of the above direct limit determine the partial quotients of α . Whilst the isomorphism

classes of the first Čech cohomology groups do not distinguish these tiling spaces, their order structure, which is determined by the above direct limit, is a rich invariant.

4.5.5. Solenoids as Hierarchical Tilings. Let S_k^d be the periodic tiling of \mathbb{R}^d of hypercubes with side-length $k \in \mathbb{N}$ and vertices on the lattice $k\mathbb{Z}^d$. These tilings define cellular decompositions \mathcal{S}_k^d of \mathbb{R}^d . Define a partial ordering on \mathbb{N} by setting m less than or equal to n if and only if m divides n ; note that $(\mathbb{N}, |)$ is a directed set. Comparing patches via translation, we have that two cells $a, b \in \mathcal{S}_1^d$ are equivalent in the tiling S_k^d (to any given radius n) if and only if a and b are related by a vector of $k\mathbb{Z}^d$. Then, with the partial order on \mathbb{N} given above, this defines a hierarchical system of tilings. Indeed, for $m | n$ we have that local patches of S_n^d determine those of S_m^d by subdividing each of the hypercubes of side-length n into $(n/m)^d$ hypercubes of side-length m .

Define \mathfrak{S}_∞^d to be the SIS associated to this hierarchical system of tilings, where we compare patches via translation. Then a Borel–Moore chain is PE in \mathfrak{S}_∞^d if and only if it is invariant under translation by some full-rank sublattice. For $d = 1$, for example, it is easy to see that $H_0(\mathfrak{S}_\infty^1) \cong \mathbb{Q}$, the isomorphism identifies $p/q \in \mathbb{Q}$ with the homology class of 0-chain which evaluates to p on each vertex of the tiling S_q^1 of intervals of length q . The group $H_1(\mathfrak{S}_\infty^1)$ is freely generated by a fundamental class for \mathbb{R}^1 . So for this example:

$$\begin{aligned}\check{H}^0(\Omega(\mathfrak{S}_\infty^1)) &\cong H^0(\mathfrak{S}_\infty^1) \cong H_1(\mathfrak{S}_\infty^1) \cong \mathbb{Z} \\ \check{H}^1(\Omega(\mathfrak{S}_\infty^1)) &\cong H^1(\mathfrak{S}_\infty^1) \cong H_0(\mathfrak{S}_\infty^1) \cong \mathbb{Q}\end{aligned}$$

The tiling space $\Omega(\mathfrak{S}_\infty^1)$ is the inverse limit of circles S^1 over the directed set $(\mathbb{N}, |)$, where for $m | n$ the map $\pi_{m,n}: S^1 \rightarrow S^1$ is given by the degree (n/m) covering map. Note that the sequence $n_i := i!$ is linearly ordered and cofinal in $(\mathbb{N}, |)$, so:

$$\Omega(\mathfrak{S}_\infty^1) \cong \varprojlim (S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 3} S^1 \xleftarrow{\times 4} S^1 \xleftarrow{\times 5} \dots)$$

Restricting the constructions above to the sequence $n_i := 2^i$, we realise Example 3.9. In this case, the tiling space is homeomorphic to

$$\mathbb{D}_2^1 := \varprojlim (S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} \dots),$$

the dyadic solenoid, and a Borel–Moore chain of \mathbb{R}^1 is PE in this setup if and only if it is periodic with some period 2^k for $k \in \mathbb{N}$.

4.5.6. Bowers and Stephenson’s “Regular” Pentagonal Tilings. Our final example will be a non-Euclidean one. Bowers and Stephenson introduced in [9] a combinatorial substitution of a pentagons which, analogously to the Euclidean case, through repeated iteration produces larger and larger patches of tiles. These finite patches may be used to define combinatorial tilings with a certain hierarchical structure. Of course, there is no tiling of Euclidean space by regular pentagons. However, one may define a supporting metric space for which each 2-cell is isometric to a regular Euclidean pentagon. The resulting tilings are supported on spaces conformally equivalent to the complex plane.

Despite not being tilings of Euclidean space, our methods here are essentially blind to the distinction, and will be as applicable to this example as to any Euclidean substitution tiling. A Bowers–Stephenson pentagonal tiling defines a CW poset \mathcal{T} , and

for each such tiling there is a uniquely defined ‘supertiling’ which subdivides to it; the substitution is recognisable.

There is no natural notion of translation on the supporting spaces of these tilings, although there is of orientation, so we shall define \mathfrak{T} by comparing patches of tiles using cellular isomorphisms which preserve the orientations of the 2-cells. There exist self-similarities of star-patches which are non-trivial on the source cells i.e., 2-cells have 5-fold rotational symmetry and the 1-cells have local 2-fold symmetry. So to compute homology using the original CW decomposition of the tiling, the method will only work in general over divisible coefficients. To compute over \mathbb{R} coefficients, we note that there are two equivalence classes of 0-cells (associated to valence 3 and valence 4 vertices), there is one equivalence class of 1-cell and one equivalence class of 2-cell. However, the 1-cells possess local symmetries reversing orientations on the 1-cells, so the approximant complexes over \mathbb{R} coefficients are

$$0 \leftarrow \mathbb{R}^2 \xleftarrow{\partial_1} 0 \xleftarrow{\partial_2} \mathbb{R} \leftarrow 0.$$

It follows that the approximant homologies over \mathbb{R} coefficients are $H_0 A_i(\mathfrak{T}; \mathbb{R}) \cong \mathbb{R}^2$, 0, \mathbb{R} for $i = 0, 1, 2$, respectively. We have that $H_2(\mathfrak{T}; \mathbb{R}) \cong \mathbb{R}$ is generated over \mathbb{R} by a fundamental class, and it follows from the approximant homologies in degree one being trivial that $H_1(\mathfrak{T}; \mathbb{R}) \cong 0$. Using the procedure outlined in §4.3 one finds the connecting maps in degree zero to be isomorphisms, so $H_0(\mathfrak{T}; \mathbb{R}) \cong \mathbb{R}^2$.

To compute homology over integral coefficients, we pass to a barycentric subdivision of the setup. Now we have four vertex types: two of them corresponding to the two vertex types of the original CW decomposition \mathcal{T} , one corresponding to the barycentre of each edge and one corresponding to the barycentre of each pentagon. There are three edge types and two face types. So the approximant chain complexes over \mathbb{Z} coefficients are

$$0 \leftarrow \mathbb{Z}^4 \xleftarrow{\partial_1} \mathbb{Z}^3 \xleftarrow{\partial_2} \mathbb{Z}^2 \leftarrow 0.$$

One computes that $H_0 A_i \cong \mathbb{Z}^2$, $H_1 A_i \cong 0$ and $H_2 A_i \cong \mathbb{Z}$. So $H_1(\mathfrak{T}_\Delta) \cong 0$, and of course $H_2(\mathfrak{T}_\Delta) \cong \mathbb{Z}$ is generated by a fundamental class. One may calculate the connecting map in degree zero as having eigenvectors which span \mathbb{Z}^2 and have eigenvalues 1 and 6, and so $H_0(\mathfrak{T}_\Delta) \cong \mathbb{Z} \oplus \mathbb{Z}[1/6]$.

To compute the PE cohomology of \mathfrak{T}_Δ (and hence the Čech cohomology of the associated tiling space), one may use the modified PE chain complexes and implement Poincaré duality. At the approximant stage, this amounts to using instead the chain complexes

$$0 \leftarrow 2\mathbb{Z} \oplus 3\mathbb{Z} \oplus 4\mathbb{Z} \oplus 5\mathbb{Z} \xleftarrow{\partial_1} \mathbb{Z}^3 \xleftarrow{\partial_2} \mathbb{Z}^2 \leftarrow 0$$

since the vertices of \mathcal{T}_Δ possess local isotropy of orders 2, 3, 4 and 5. After computing the connecting maps and corresponding direct limits, we obtain:

$$\begin{aligned} \check{H}^0(\Omega(\mathfrak{T}_\Delta)) &\cong H^0(\mathfrak{T}_\Delta) \cong H_2^\dagger(\mathfrak{T}_\Delta) \cong \mathbb{Z} \\ \check{H}^1(\Omega(\mathfrak{T}_\Delta)) &\cong H^1(\mathfrak{T}_\Delta) \cong H_1^\dagger(\mathfrak{T}_\Delta) \cong 0 \\ \check{H}^2(\Omega(\mathfrak{T}_\Delta)) &\cong H^2(\mathfrak{T}_\Delta) \cong H_0^\dagger(\mathfrak{T}_\Delta) \cong \mathbb{Z} \oplus \mathbb{Z}[1/6] \end{aligned}$$

The space $\Omega(\mathfrak{T}_\Delta)$ corresponds to the moduli space of pointed regular pentagonal tilings taken modulo rotational symmetries, the analogue of the space $\Omega^0 = \Omega^{\text{rot}}/\text{SO}(d)$ (see [4]) associated to a Euclidean tiling.

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